PART II (γ *version*): *The ring* $\mathcal{A}^G$ *and the Lie algebra* $\mathcal{A}^G_\rho$

Abstract: *We study an action of the group of permutations of the positive integers in the ring of arithmetic functions and the induced analytic representations. The Lie algebra of the derivations on the ring of invariant arithmetic functions displays a Witt-like structure and we study the corresponding structure of the analytic representation.*

§1. INTRODUCTION

The “ring $\mathcal{A}$ of arithmetic functions” is the ring of complex functions $a:\mathbb{N}\to\mathbb{C}$ with pointwise addition and Dirichlet convolution as product:

$$(a*b)(n)=\sum_{d|n} a(d)b\left(\frac{n}{d}\right) \qquad (1.1)$$

(The pointwise multiplication will be indicated *ab*, as usual). It is of course a complex algebra and its structure has been studied extensively, v. gr. in [C-E] and [HNS]. It is an associative commutative *UFD*; the invertible elements are the functions $a : \mathbb{N} \to \mathbb{C}$ such that $a(1) \neq 0$ so that it is a local ring; it is not noetherian and in fact it has an infinite strictly increasing chain of prime ideals, so its Krull dimension is infinite. (We gave an example of such a chain in Part I, because we have not found any reference in the literature).

On the other hand, as we know, the key technique of the analytic number theory is what H. N. Shapiro in [HNS] calls "analytic representations" of $\mathcal{A}$, i. e.: ring morphisms $\Phi : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is an algebra of holomorphic functions defined on some domain. The classical one is given by the Dirichlet series $\Phi(a)(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$. In [HNS] the author study the derivations in $\mathcal{A}$ and their relationship with additive functions and the analytic representations, imposing some extra conditions on the ring morphisms.

In this work we define an action of the group $G = S_\infty$ of permutations in the set on positive integers in the ring $\mathcal{A}$. This action preserves the ring structure, so we have a subring $\mathcal{A}^G$ of invariant (or *symmetric*) arithmetic functions. Many important arithmetic functions are, in fact, symmetric. The ring $\mathcal{A}$ is a complex algebra and the module of continuous derivations (in the sense of Shapiro) on this ring is characterized in [HSN]. It not difficult to see that there is an essentially unique invariant derivation (i.e, a derivations that stabilizes the subalgebra $\mathcal{A}^G$), so that the module of continuous derivations in this algebra is free of rank one. Then, there is a natural Lie algebra structure in $\mathcal{A}^G$. The aim of the present work is the study of this structure. On the other hand, there is a promising line of work to do in the study of the action of S_∞ on $\mathcal{A}$. The infinite symmetric group $S(\infty)$ (permutations of finite support) is a dense subgroup of S_∞ and this group and its representations are the subject of intense study in the past decades (v.gr. [Ou])

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§2. THE ACTION OF AND THE FIXED RING.

Let $\mathbb{P} = \{p_1, p_2, ...\}$ be the set of positive prime numbers given in some fixed order (v. gr. the natural one). Let G be the group of permutations $\gamma : \mathbb{N} \to \mathbb{N}$ of the positive integers. For each positive integer $n = \prod_{k=1}^{\infty} p_k^{v_{p_k}(n)}$ (where each v_p is the p-adic valuation) and each $\gamma \in G$ let us define

$$\gamma.n = \prod_{k=1}^{\infty} p_{\gamma^{-1}(k)}^{v_{p_k}(n)} \tag{2.1}$$

Observe that $\gamma.1 = 1$ for all $\gamma \in G$ (all p-adic valuations vanish in 1). Now, given a function, $a : \mathbb{N} \to \mathbb{C}$ for each $\gamma \in G$ let us define

$$a^{\gamma} : \mathbb{N} \to \mathbb{C}, \quad n \mapsto a(\gamma.n) \tag{2.2}$$

PROP. 2.1:

(*i*) The setting (2.1) and $\gamma.1 = 1$ define a left action of G in the set $\mathbb{N}$ of positive integers.

(*ii*) The setting (2.2) defines a right action of G in the ring $\mathbb{A}$. In particular, this means that

$$(a*b)^{\gamma} = a^{\gamma} * b^{\gamma} \tag{2.3}$$

Proof of (2.3): For each $\alpha \in G$, n and m in $\mathbb{N}$, a and b in $\mathbb{A}$:

(a) $\alpha.(nm) = (\alpha.n)(\alpha.m)$:

$$\alpha.(nm) = \alpha.\left(\prod_{k=1}^{\infty} p_k^{v_{p_k}(n)+v_{p_k}(m)}\right) = \prod_{k=1}^{\infty} p_{\alpha^{-1}(k)}^{v_{p_k}(n)+v_{p_k}(m)} = \left(\prod_{k=1}^{\infty} p_{\alpha^{-1}(k)}^{v_{p_k}(n)}\right)\left(\prod_{k=1}^{\infty} p_{\alpha^{-1}(k)}^{v_{p_k}(m)}\right) = (\alpha.n)(\alpha.m)$$

(b) $d|n \Leftrightarrow \alpha(d)|\alpha(n)$, because $d = \prod_{k=1}^{\infty} p_k^{v_{p_k}(d)}$ is a divisor of $n = \prod_{k=1}^{\infty} p_k^{v_{p_k}(n)}$ iff $v_{p_k}(d) \le v_{p_k}(n)$ for all k iff $\alpha.d = \prod_{k=1}^{\infty} p_{\alpha^{-1}(k)}^{v_{p_k}(d)}$ is a divisor of $\alpha.n = \prod_{k=1}^{\infty} p_{\alpha^{-1}(k)}^{v_{p_k}(n)}$

(c)

$$(a^\alpha * b^\alpha)(n) = \sum_{d|n} a^\alpha(d) b^\alpha\left(\frac{n}{d}\right) = \sum_{d|n} a(\alpha.d) b\left(\alpha.\frac{n}{d}\right) \stackrel{(a)}{=} \sum_{d|n} a(\alpha.d) b\left(\frac{\alpha.n}{\alpha.d}\right) \stackrel{(b)}{=}$$

$$= \sum_{d'|\alpha(n)} a(d') b\left(\frac{\alpha.n}{d'}\right) = (a*b)(\alpha.n) = (a*b)^\alpha(n)$$

□

We have, thus, a subring (in fact a complex subalgebra) of $\mathbb{A}$, the ring of invariant arithmetic functions:

$$\mathbb{A}^G = \left\{ a \in \mathbb{A} / \forall \in G : a^\gamma = a \right\} \qquad (2.4)$$

This ring contains many important arithmetic functions, like the Moebius and Liouville functions. Of course it contains also the constant functions, and formula (2.3) implies that the Dirichlet convolutions $d_0 = 1 * 1$ is also an invariant function.

In order to find a basis of A^G (as linear complex space), we have to observe the quotient $\mathbb{N}/G$ and find the characteristic functions of the orbits. For this purpose, we introduce some definitions and notations.

Let Λ be the set of infinite sequences of nonnegative integers $\breve{v} = (v_1, v_2, \ldots, v_k, \ldots)$ such that

(a) There exists a nonnegative integer r such that $v_{r+1} = v_{r+2} = \ldots = v_k = \ldots = 0$

(b) $v_1 \geq v_2 \geq ... \geq v_r \geq 1$

(This is the set Λ of partitions of the positive integers). The number r in (a) will be called "the order" of $\breve{v}=(v_1,v_2,...,v_k,...)$ and will be denoted $o(\breve{v})$. Observe that :

(1) $o(\breve{v})=0 \Leftrightarrow \breve{v}=\breve{0}=(0,0,0,...)$

(2) Λ is a commutative monoid with respect to the usual addition of sequences.

(3) For each integer $n \geq 2$ there is a unique $\breve{v}\in\Lambda$ and $o(\breve{v})=r$ distinct primes $q_1, q_2,..., q_r$ such that $n = q_1^{v_1} q_2^{v_2} ... q_r^{v_r}$: instead of the usual ordering of the prime factors , just use the decreasing ordering $v_1 \geq v_2 \geq ... \geq v_r \geq 1$ of the p-adic valuations.

Finally, let us consider the function $\Lambda \to \mathbb{N}$, $\breve{v} \mapsto n_{\breve{v}}$ such that

$$n_{\breve{v}} = p_1^{v_1} p_2^{v_2} ... p_r^{v_r} \tag{2.5}$$

where $\breve{v}=(v_1,v_2,...,v_r,0,0,...)\in\Lambda$ has order r and $p_1, p_2,..., p_r$ are the first r primes (given in the fixed chosen ordering). Observe that $n_{\breve{0}}=1$

***Notation* 2.1**: It is natural to denote $\breve{v}=(v_1,v_2,...,v_r)$ an element $\breve{v}=(v_1,v_2,...,v_r,0,0,...)\in\Lambda$ of order r, i.e.: it is assumed that the last component indicated is the last (and least) non zero component (an exception is imposed with $\breve{0}$). This is a simplification of notation for the following work, except for the sum $\breve{v}+\breve{w}$.

***Notation* 2.2**: For each $\breve{v}=(v_1,v_2,...,v_r,0,0,...)\in\Lambda$, let us denote $\dot{v}=v_1+v_2+...+v_r$ (it may be called the " trace" of $\breve{v}$). Observe that for each positive integer n, $\{\breve{v}\in\Lambda : \dot{v}=n\}$ is the set of partitions of n.

PROP. 2.2:

(i) For each $n\in\mathbb{N}$ there is a unique $\breve{v}\in\Lambda$ such that $G.n=G.n_{\breve{v}}$.

(ii) The mapping $\Lambda \to \mathbb{N}/G$, $\breve{v}\mapsto G.n_{\breve{v}}$ is a bijection.

(iii) For each $\breve{v}=(v_1,v_2,...,v_r,0,0...)\in\Lambda : G.n_{\breve{v}} = \left\{ q_1^{v_1} q_2^{v_2} ... q_r^{v_r} : \overbrace{q_1,q_2,...,q_r}^{distinct} \in \mathbb{P} \right\}$ (recall that $v_1 \geq v_2 \geq ... \geq v_r \geq 1$ and observe that $G.n_{\breve{0}}=G.1=\{1\}$)

Proof: Almost trivial: what we did is to choose a section of the canonical surjection $\mathbb{N} \to \mathbb{N}/G$ and we labeled the orbits with Λ . □

***Remark* 2.1:** For each fixed positive integer *m*, the number of distinct orbits $G.n_{\breve{v}}$ of order $\dot{v} = m$ is *p*(*m*), the number of its partitions. This number grows fast with *m*. More precisely, it is well known (since Hardy and Ramanujan) that for large *m*: $p(m) \approx \dfrac{e^{\pi\sqrt{\frac{2}{3}m}}}{4m\sqrt{3}}$.

Now, for each $\breve{v} \in \Lambda$ let us denote $X_{\breve{v}} : \mathbb{N} \to \{0,1\}$ the characteristic function of the orbit $G.n_{\breve{v}}$. Of course, these functions are invariant, by definition. Observe that $X_{\breve{0}} = e_1$ (we use here the symbols e_n for the functions $e_n(m) = \delta_{mn}$, because there will be a lot of deltas in the next sections).

Now, we prove that the set $\{X_{\breve{v}} : \breve{v} \in \Lambda\}$ is a linear basis of $\mathcal{A}_\rho^G$ (in a sense explained below).

PROP. 2.3:

(*i*) The set $\{X_{\breve{v}} : \breve{v} \in \Lambda\}$ is linearly independent (over $\mathbb{C}$);

(*ii*) For each $a \in \mathcal{A}^G : a = \sum_{\breve{v} \in \Lambda} a(n_{\breve{v}}) X_{\breve{v}}$, in the following sense: for each $n \in \mathbb{N}$, the sum $\sum_{\breve{v} \in \Lambda} a(n_{\breve{v}}) X_{\breve{v}}(n)$ is finite (furthermore: it has just one term, because each *n* belongs to just one orbit) and $a(n) = \sum_{\breve{v} \in \Lambda} a(n_{\breve{v}}) X_{\breve{v}}(n)$.

Proof of (*i*): $\sum_{i=1}^{m} c_i X_{\breve{v}_i} = 0 \Rightarrow \sum_{i=1}^{m} c_i \overbrace{X_{\breve{v}_i}(n_{\breve{v}_j})}^{\delta_{ij}} = 0 \Rightarrow c_j = 0$.

Proof of (*ii*): Given $n \in \mathbb{N}$ let $\breve{v}_0 \in \Lambda$ be the unique $\breve{v} \in \Lambda$ such that $G.n = G.n_{\breve{v}_0}$. Then

$\sum_{\breve{v} \in \Lambda} a(n_{\breve{v}}) X_{\breve{v}}(n) = a(n_{\breve{v}_0}) = a(n)$.□

PROP. 2.4. For each $\breve{v}, \breve{w} \in \Lambda$:

$$X_{\breve{v}} * X_{\breve{w}} = \sum_{\breve{u} \in \Lambda(\breve{v},\breve{w})} \delta_{\dot{v}+\dot{w}}^{\dot{u}} X_{\breve{u}} \qquad (2.6)$$

where

$$\Lambda(\breve{v},\breve{w}) = \{\breve{u} \in \Lambda : \max\{o(\breve{v}), o(\breve{w})\} \le o(\breve{u}) \le o(\breve{v}) + o(\breve{w})\}$$

Proof: Let r be the order of $\breve{v}$ and let s be the order of $\breve{w}$. A term $X_{\breve{v}}(d)X_w(d')$ of the convolution $(X_{\breve{v}} * X_{\breve{w}})(n) = \sum_{dd'=n} X_{\breve{v}}(d)X_{\breve{w}}(d')$ does not vanish iff the divisors d and $d' = \frac{n}{d}$ of n are of the form $d = q_1^{v_1}q_2^{v_2}...q_r^{v_r}$ (with distinct prime factors q) and $d' = q'^{w_1}_1 q'^{w_2}_2 ... q'^{w_s}_s$ (with distinct prime factors q'). Then $X_{\breve{v}}(d)X_w(d') \neq 0$ iff n is of the form $n = q_1^{v_1}q_2^{v_2}..q_r^{v_r}\, q'^{w_1}_1 q'^{w_2}_2 .. q'^{w_s}_s$ All this n are in the orbits $G.n_{\breve{u}}$, where $n_{\breve{u}} = p_1^{u_1}p_2^{u_2}...p_t^{u_t}$, with $\dot{u} = \dot{v} + \dot{w}$: if $q_i = q_j'$, then there is a prime factor $p_k^{v_i+w_j}$ of $n_{\breve{u}}$. Now, assuming that $r \geq s$ (without losing generality), it is clear that $o(\breve{u}) \geq r$, with equality in the case that $\{q'_1,...,q'_s\} \subseteq \{q_1,...,q_r\}$; on the other hand, it is apparent that $o(\breve{u}) \leq r + s$, with equality in the case $\{q'_1,...,q'_s\} \cap \{q_1,...,q_r\} = \emptyset$. □

***Remark* 2.2**: The formula (2.6) is not the same as the formula of the product of Schur polynomials, so we do not think that manipulations of Young tableaux may help in our case.

***Remark* 2.3**: There is something to do: extend the action to the quotient field $\mathcal{Q}(\mathcal{A})$ and study $\mathcal{Q}(\mathcal{A}^G)$, $\mathcal{Q}(\mathcal{A})^G$ and the field extension $\mathcal{Q}(\mathcal{A}) / \mathcal{Q}(\mathcal{A})^G$.

§3. INVARIANT DERIVATIONS AND THE LIE ALGEBRA $\mathcal{A}_\rho^G$

A derivation $\partial : A^G \to A^G$ is a $\mathbb{C}$-linear mapping such that $\partial(a * b) = \partial a * b + a * \partial b$ for all a and b in $\mathcal{A}^G$. Basic examples of derivations $\partial : A \to A$ are the mappings of the form $\partial_l a = la$ (pointwise multiplication), where $l \in \mathcal{A}$ is a completely additive function (i.e $l(mn) = l(m) + l(n)$; against the number theoretic tradition, we would prefer the name "logarithmic", for obvious reasons). The basic examples are the p-adic valuations v_p and log. Observe that if $l : \mathbb{N} \to \mathbb{C}$ is a logarithmic function, then $l(1) = 0$ (because $l(1) = l(1) + l(1)$) and that for each $n = \prod_{k=1}^{\infty} p_k^{v_{p_k}(n)} \geq 2$ (it is a finite product), $l(n) = \sum_k l(p_k)v_{p_k}(n)$ (finite sum). That is to say: $l = \sum_k l(p_k)v_{p_k}$ (in the sense of "pointwise-finite sum convergence"). Shapiro, in [HNS], give a characterization of the continuous derivations $\partial : A \to A$ in the hole ring $\mathcal{A}$, where the continuity is considered with respect to a valuation norm. Essentially, these derivations are of the form $\sum_k a_k * \partial_{l_k}$, where $a_k \in \mathcal{A}, \partial_{l_k} a = l_k a$ and each $l_k \in \mathcal{A}$ is a logarithmic function. In order to obtain invariant derivations, i.e., derivations $\partial : A^G \to A^G$, we look at the derivations $\partial = \sum_k a_k * \partial_{l_k}$ where $a_k \in \mathcal{A}^G$ and each $l_k \in \mathcal{A}$ is an invariant logarithmic function. But if $l = \sum_k l(p_k) v_{p_k}$ is an invariant logarithmic function, then it is constant on the set $\mathcal{P}$ of positive primes (for each positive prime p, it is clear that $G.p = \mathcal{P}$), then necessarily $l = c \sum_k v_{p_k} = c\rho$, where

$$\rho : \mathbb{N} \to \mathbb{N} \ , \ n \mapsto \sum_{n=1}^{\infty} v_p(n) \tag{3.1}$$

(it is a finite sum) is the number of prime factors of n, counting the multiplicities. In [Shapiro]. (page 309) it is denoted $\Omega(n)$ and will play an essential role in the following. Then, the invariant continuous (in the sense of "pointwise finite sum convergence") derivations are of the form $a * \partial_\rho$, $a \in \mathcal{A}^G$ that is to say, $Der_{cont}(\mathcal{A}^G, \mathcal{A}^G)$ is the free $\mathcal{A}^G$ -module of rank 1 with basis $\{\partial_\rho\}$. For the Lie bracket we have to compute:

$$[a_1 * \partial_\rho, a_2 * \partial_\rho](b) = (a_1 * \partial_\rho)(a_2 * \rho b) - (a_2 * \partial_\rho)(a_1 * \rho b) =$$

$$= (a_1 * \partial_\rho a_2) * \rho b + a_2 * (a_1 * \partial_\rho(\rho b)) - (a_2 * \partial_\rho a_1) * \rho b - a_1 * (a_2 * \partial_\rho(\rho b)) =$$

$$= (a_1 * \rho a_2) * \rho b + a_2 * (a_1 * \rho^2 b) - (a_2 * \rho a_1) * \rho b - a_1 * (a_2 * \rho^2 b) = (\text{* is associative})$$

$$= a_1 * \rho a_2 * \rho b + a_2 * a_1 * \rho^2 b - a_2 * \rho a_1 * \rho b - a_1 * a_2 * \rho^2 b = (\text{* is commutative})$$

$$= a_1 * \rho a_2 * \rho b - a_2 * \rho a_1 * \rho b = (a_1 * \rho a_2 - a_2 * \rho a_1) * \rho b = \{(a_1 * \rho a_2 - a_2 * \rho a_1) * \partial_\rho\} b$$

That is to say:

$$[a_1 * \partial_\rho, a_2 * \partial_\rho] = (a_1 * \rho a_2 - a_2 * \rho a_1) * \partial_\rho \tag{3.2}$$

Then, we have a Lie algebra structure in $\mathcal{A}^G$ given by the bracket

$$[a_1, a_2] = a_1 * \rho a_2 - a_2 * \rho a_1 \ . \tag{3.3}$$

This Lie algebra structure will be denoted $\mathcal{A}^G_\rho$ and - of course - can be defined without any mention to the derivations. But the motivation of the formula (3.3) is clearly attached to the fact that $Der_{cont}(\mathcal{A}^G,\mathcal{A}^G)\cong\mathcal{A}^G$(as $\mathcal{A}^G$ -modules). We close this section with a formula that has a "Witt flavor".

***Remark* 3.1:** Two useful identities:

1) $\rho.(a*b)=(\rho.a)*b+a*(\rho.b)$

2) $\rho.[a,b]=[\rho.a,b]+[a,\rho.b]=a*(\rho^2.b)-(\rho^2.a)*b$

PROPOSITION 3.1: For each $\breve{v}=(v_1,v_2,...,v_r,0,0,..)\in\Lambda$, $\breve{w}=(w_1,w_2,...,w_s,0,0,..)\in\Lambda$:

$$[X_{\breve{v}},X_{\breve{w}}]=(\dot{w}-\dot{v})X_{\breve{v}}*X_{\breve{w}} \tag{3.4}$$

Proof : For each n : $\rho(n)X_{\breve{v}}(n)=\rho(n_{\breve{v}})X_{\breve{v}}(n)=\dot{v}X_{\breve{v}}(n)$, so that $(\rho X_{\breve{v}})*X_{\breve{w}}=\dot{v}X_{\breve{v}}*X_{\breve{w}}$. Then, $[X_{\breve{v}},X_{\breve{w}}]=X_{\breve{v}}*(\rho X_{\breve{w}})-(\rho X_{\breve{v}})*X_{\breve{w}}=(\dot{w}-\dot{v})X_{\breve{v}}*X_{\breve{w}}$ □

For $\breve{v}=\breve{0}$ we have $[X_{\breve{0}},X_{\breve{w}}]=\dot{w}X_{\breve{0}}*X_{\breve{w}}=\dot{w}e_1*X_{\breve{w}}=\dot{w}X_{\breve{w}}$. Then, it is natural (and standard) to consider the subspaces

$$L_\lambda=\{a\in\mathcal{A}_\rho^G:[X_{\breve{0}},a]=\lambda a\} \tag{3.5}$$

(for each complex λ). Then, we have the (also standard) facts:

PROPOSITION 3.2: For each

(i) $L_\lambda\neq\{0\}\Leftrightarrow\lambda$ is a nonnegative integer and for each nonnegative integer m, L_m is finite dimensional. Furthermore: $L_m=\bigoplus_{\dot{u}=m}\mathbb{C}X_{\breve{u}}$ (in particular: $L_0=\mathbb{C}X_{\breve{0}}=\mathbb{C}e_1$)

(ii) $[L_m,L_n]\subseteq L_{m+n}$

(iii)[1] $\mathcal{A}^G=\overline{\bigoplus_{m\geq 0}L_m}$

[1] This expression means that each $a\in\mathcal{A}^G$ admits a unique decomposition as a sum $a=\sum_m a_m$ of homogeneous terms, such that for each n the sum $\sum_m a_m(n)$ is finite (and equals $a(n)$, of course). Now, (3.4) implies trivially that if $\breve{v}$ and $\breve{w}$ verify $\dot{v}=\dot{w}$, then $[X_{\breve{v}},X_{\breve{w}}]=0$. Then,

(iii) is a decomposition of the Lie algebra $\mathcal{A}^G$ as a sum of finite dimensional abelian subalgebras.

P/ (i) Given $a=\sum_{\breve{v}\in\Lambda} a(n_{\breve{v}})X_{\breve{v}} \in \mathcal{A}_\rho^G$, $[X_{\breve{0}},a]=\sum_{\breve{v}\in\Lambda} \dot{v}a(n_{\breve{v}})X_{\breve{v}}$. Then $[X_{\breve{0}},a]=\lambda a \Leftrightarrow$ for all $\breve{v}\in\Lambda$: $\dot{v}a(n_{\breve{v}})=\lambda a(n_{\breve{v}})$. If $a\neq 0$, some $a(n_{\breve{v}_0})\neq 0$ and thus $\lambda=\dot{v}_0=m\in\mathbb{Z}_{\geq 0}$. It follows that for all $\breve{v}\in\Lambda$ such that $a(n_{\breve{v}})\neq 0$: $\dot{v}=m$, so that $a=\sum_{\dot{v}=m} a(n_{\breve{v}})X_{\breve{v}} \in \bigoplus_{\dot{v}=m}\mathbb{C}X_{\breve{v}}$.

P/ (ii) This is a standard fact: Given $a\in L_m$ and $b\in L_n$:

$$[X_{\breve{0}},[a,b]]=-[a,[b,X_{\breve{0}}]]-[b,[X_{\breve{0}},a]]=[a,[X_{\breve{0}},b]]+[[X_{\breve{0}},a],b]=n[a,b]+m[a,b]=(m+n)[a,b]$$

P/(iii) $\mathcal{A}_\rho^G=\bigoplus_{\breve{v}\in\Lambda}\mathbb{C}X_{\breve{v}}=\bigoplus_{m=0}^{\infty}(\bigoplus_{\dot{v}=m}\mathbb{C}X_{\breve{v}})=\bigoplus_{m=0}^{\infty}L_m$ (the upper bar is understood)□

Using this grading, we obtain the following structure facts about the Lie algebra $\mathcal{A}_\rho^G$:

PROPOSITION 3.3: In the Lie algebra $\mathcal{A}_\rho^G$, the linear subspace $\mathcal{I}=\{a\in\mathcal{A}_\rho^G : a(1)=0\}$ is a maximal ideal, it is an homogeneous ideal (with respect the above grading; recall: an ideal is homogeneous or graded if contains the homogeneous components of its elements, and recall that we are dealing with the *pointwise finite-sum convergence*) Furthermore:

(i) $\mathcal{I}=[\mathcal{A}_\rho^G,\mathcal{A}_\rho^G]$

(ii) $\mathcal{A}_\rho^G=\mathbb{C}e_1\oplus\mathcal{I}$ (Direct sum of an abelian subalgebra and an ideal)

(iii) $[e_1,\mathcal{A}_\rho^G]=\mathcal{I}$, i.e.: for each $h\in\mathcal{I}$ there is a $b\in\mathcal{A}_\rho^G$ such that $[e_1,b]=h$, and conversely for all $b\in\mathcal{A}_\rho^G$, $[e_1,b]\in\mathcal{I}$.

P/ We will make use of the following easy formulae: for all a, b in $\mathcal{A}_\rho^G$:

(1) $[a,b](1)=(a*(\rho b))(1)-((\rho a)*b)(1)=a(1)\overbrace{(\rho b)(1)}^{=0}-\overbrace{(\rho a)(1)}^{=0}b(1)=0$

(2) $[e_1,b]=e_1*(\rho b)-\overbrace{(\rho e_1)}^{=0}*b=\rho b$. For $b=X_{\breve{v}}$: $[e_1,X_{\breve{v}}]=\rho X_{\breve{v}}=\dot{v}X_{\breve{v}}$, because for each $n\in G.n_{\breve{v}}$ we have $\rho(n)=\dot{v}$. (Recall: $X_{\breve{0}}=e_1$).

Then:

(A) That $\mathcal{I}=\{a\in\mathcal{A}_\rho^G : a(1)=0\}$ is an ideal follows from the first formula.

(B) The inclusion $[\mathcal{A}_\rho^G, \mathcal{A}_\rho^G] \subseteq \mathcal{I}$ is another straightforward consequence of the same formula.

(B) $\mathcal{I} \subseteq [\mathcal{A}_\rho^G, \mathcal{A}_\rho^G]$: given $h \in \mathcal{I}$ (i.e. $h(1) = 0$), let $b \in \mathcal{A}_\rho^G$ be the arithmetic function such that $b(n) = \dfrac{h(n)}{\rho(n)}$ if $n \geq 2$ and $b(1) = 0$ (it could be defined by any other value in 1, so b is not unique). Observe that b is, indeed, invariant: $b(\gamma.n) = \dfrac{h(\gamma.n)}{\rho(\gamma.n)} = \dfrac{h(n)}{\rho(n)} = b(n)$ para $n \geq 2$ and $b(\gamma.1) = b(1)$. Now, it is obvious that $[e_1, b] = \rho b = h$.

(C) $\mathcal{A}_\rho^G = \mathbb{C}e_1 \oplus \mathcal{I}$:

To prove this, let us consider (it is the standard way) the projection $\pi : \mathcal{A}_\rho^G \to \mathcal{A}_\rho^G$ such that $\pi(a) = a - a(1)e_1$ (easy: $\pi(\pi(a)) = \pi(a) - \overbrace{\pi(a)(1)}^{=0}e_1 = \pi(a)$). Then, $\pi(\mathcal{A}_\rho^G) = \mathcal{I}$) (easy: $\pi(a) = a \Leftrightarrow a(1) = 0$) and $Ker(\pi) = \mathbb{C}e_1$ (easy: $\pi(a) = 0 \Leftrightarrow a = a(1)e_1$).

(D) $\mathcal{I}$ is maximal: let $\mathcal{J}$ be an ideal in $\mathcal{A}_\rho^G$ such that $\mathcal{I} \subseteq \mathcal{J}$. From (C) each $a \in \mathcal{A}_\rho^G$ is of the form $a = \lambda e_1 + a_\mathcal{I}$ for a unique $\lambda \in \mathbb{C}$ and a unique $a_\mathcal{I} \in \mathcal{I}$. Then, if $e_1 \in \mathcal{J}$, we have that $\mathcal{J} = \mathcal{A}_\rho^G$, because each $a = \lambda e_1 + a_\mathcal{I} \in \mathcal{A}_\rho^G$ is a sum of $\lambda e_1 \in \mathcal{J}$ plus an $a_\mathcal{I} \in \mathcal{I} \subseteq \mathcal{J}$; if $e_1 \notin \mathcal{J}$, every $a = \lambda e_1 + a_\mathcal{I} \in \mathcal{J}$ verifies $\lambda = a(1) = 0$ (if for some $a_0 = \lambda_0 e_1 + a_{0\mathcal{I}} \in \mathcal{J}$ is $\lambda_0 \neq 0$, then $e_1 = \overbrace{\dfrac{1}{\lambda_0} a_0}^{\in \mathcal{J}} - \overbrace{\dfrac{1}{\lambda_0} a_{0\mathcal{I}}}^{\in \mathcal{I} \subseteq \mathcal{J}} \in \mathcal{J}$). That is to say: $\mathcal{J} \subseteq \mathcal{I}$.

(E) $[e_1, \mathcal{A}_\rho^G] = \mathcal{I}$: The inclusion $[e_1, \mathcal{A}_\rho^G] \subseteq \mathcal{I}$ is, again, a consequence of formula (1): for all b, $[e_1, b] = \rho b$ vanishes in 1. Conversely, all $h \in \mathcal{A}_\rho^G$ such that $h(1) = 0$ is of the form $h = [e_1, b]$ for some $b \in \mathcal{A}_\rho^G$ (see proof of (B)).

(F) $\mathcal{I}$ is the unique homogeous maximal ideal: that $\mathcal{I}$ is homogeneous follows from its own definition: each $a \in \mathcal{I}$ is of the form $a = \sum_{\breve{v} \in \Lambda - \{\breve{0}\}} a(n_{\breve{v}}) X_{\breve{v}}$, and each $a(n_{\breve{v}}) X_{\breve{v}}$, with $\breve{v} \neq \breve{0}$, belongs to $\mathcal{I}$ (here, an ideal is supposed to be closed under *pointwise finite-sums*). Now, let $\mathcal{J}$ be a non trivial homogeneous ideal in $\mathcal{A}_\rho^G$. If $e_1 \in \mathcal{J}$, then $\mathcal{I} \overset{(E)}{=} [e_1, \mathcal{A}_\rho^G] \overset{e_1 \in \mathcal{J}}{\subseteq} \mathcal{J}$. But $\mathcal{I}$ is maximal, so that $\mathcal{I} = \mathcal{J}$, which is absurd, because $e_1 \notin \mathcal{I}$ (or $\mathcal{J} = \mathcal{A}_\rho^G$, which is not the case: we are dealing with a not trivial ideal). Now, given $a = a(1)X_{\breve{0}} + \sum_{\breve{v} \in \Lambda - \{\breve{0}\}} a(n_{\breve{v}}) X_{\breve{v}} \in \mathcal{J}$ we have necessarily $a(1)=0$: $\mathcal{J}$ is homogeneous, then $a(1)X_{\bar{0}} = a(1)e_1 \in \mathcal{J}$, and if $a(1) \neq 0$ we would have $e_1 = \dfrac{1}{a(1)} a(1)e_1 \in \mathcal{J}$, which is absurd. That is to say: all $a \in \mathcal{J}$ verifies $a(1) = 0$, so $\mathcal{J} \subseteq \mathcal{I}$

□

COROLLARY 3.4: Let $\mathcal{J}$ be an ideal in $\mathcal{A}_\rho^G$. If $e_1 \in \mathcal{J}$, then $\mathcal{J} = \mathcal{A}_\rho^G$.

P/ It is part of part of the preceding proof: if $e_1 \in \mathcal{J}$, then $\mathcal{I} \overset{(iii)}{=} [e_1, \mathcal{A}_\rho^G] \overset{e_1 \in \mathcal{J}}{\subseteq} \mathcal{J}$. But $\mathcal{I}$ is maximal, so $\mathcal{J} = \mathcal{I}$ or $\mathcal{J} = \mathcal{A}_\rho^G$. But the first case is not possible, because $e_1 \notin \mathcal{I}$. □

PROPOSITION 3.5. $\mathcal{Z}(\mathcal{A}_\rho^G) = \{0\}$

P/ Let $b \in \mathcal{A}_\rho^G$ such that $[a,b] = 0$ for all $a \in \mathcal{A}_\rho^G$. Particularly: $[e_1, b] = \rho b = 0$, which implies that $b(n) = 0$ for all $n \geq 2$. Now, for $a = 1$ (constant):

$$0 = [1,b](2) = (b * \rho)(2) - ((\rho b) * 1)(2) = b(1)\overbrace{\rho(2)}^{=1} + \overbrace{b(2)}^{=0}\overbrace{\rho(1)}^{=0} - \overbrace{(\rho b)(1)}^{=0} = b(1)$$

□

We have proved that: $\mathcal{A}_\rho^G$ has trivial center, $\mathcal{A}_\rho^G \big/ [\mathcal{A}_\rho^G, \mathcal{A}_\rho^G]$ is one dimensional and $[\mathcal{A}_\rho^G, \mathcal{A}_\rho^G]$ is a maximal ideal, and it is the unique homogeneous maximal ideal. That is to say: $\mathcal{A}_\rho^G$ is "quasi perfect" …

***Remark* 3.6** (Not in the β-version): we have, from the results above:

$$A_\rho^G = \overbrace{[A_\rho^G, A_\rho^G]}^{Max\ ideal} \oplus \mathbb{C}e_1 \cong [A_\rho^G, A_\rho^G] \oplus \overbrace{A_\rho^G \big/ [A_\rho^G, A_\rho^G]}^{1\text{-dim}}$$

(Caution: $\mathbb{C}e_1$ is not in the center of A_ρ^G: $Z(A_\rho^G) = \{0\}$). On the other hand, $\mathcal{I} = [A_\rho^G, A_\rho^G] = \{a \in A_\rho^G : a(1) = 0\} = [e_1, A_\rho^G]$ is perfect:

Proof: Given $a \in \not I = [A_\rho^G, A_\rho^G]$, taking account of $A_\rho^G = \overbrace{[A_\rho^G, A_\rho^G]}^{\not I} \oplus \mathbb{C}e_1$ and with the obvious notations:

$$a = \sum_{i=1}^{m}[b_i, c_i] = \sum_{i=1}^{m}[b_i^{\not I} + \lambda_i e_1, c_i^{\not I} + \mu\, e_1] = \sum_{i=1}^{m}\overbrace{[b_i^{\not I}, c_i^{\not I}]}^{\in[\not I, \not I]} + \sum_{i=1}^{m}\mu_i\overbrace{[b_i^{\not I}, e_1]}^{-\rho b_i^{\not I}} + \sum_{i=1}^{m}\lambda_i\overbrace{[e_1, c_i^{\not I}]}^{\rho c_i^{\not I}} + 0$$

but $(\rho b)(1) = 0$, for all *b*, so the second member vanish at 1 and thus *a* lives in $\not I$. □

Then, $\not I$ has a universal central extension (it is a perfect Lie algebra) $\tilde{\not I} = \not I \oplus_P H_2(\not I, \mathbb{C})$ for some standard (?) "universal" cocycle $P \in Z^2(\not I, H_2(\not I, \mathbb{C}))$.

*Question*1: $H_2(\not I, \mathbb{C})$?. If it vanishes, $\not I$ may be simply connected (see "recognition criterion")

Question 2: Is $\not I$ a twisted form of some interesting algebra?

Question 3: $\not A_\rho^G \oplus H_2(\not I, \mathbb{C}) = \not I \oplus_P H_2(\not I, \mathbb{C}) \oplus \mathbb{C}e_1$?

***Remark* 3.7** (Not in the β-version): Non central extensions: for each A_ρ^G -module *M* and each cocycle $\omega \in Z^2(L, M)$ we have an extension

$$0 \to M \to \overbrace{A_\rho^G \oplus_\omega M}^{\breve{A}} \xrightarrow{\ \pi\ } A_\rho^G \to 0$$

where

$$[a + m, a' + m'\breve{]} = [a, a'\breve{]} + \overbrace{[a, m'\breve{]}}^{a.m'} + \overbrace{[m, a'.m\breve{]}}^{-l'.m} + \overbrace{[m, m']}^{=0} = \overbrace{[a, a']}^{\in A_\rho^G} + \overbrace{\omega(a, a') + a.m' - a'.m}^{\in M}$$

and (recall the standard complex)

$$(\delta^2\omega)(a, a', a'') = -\omega([a, a'], a'') + \overbrace{\omega([a, a''], a')}^{= -\omega([a'', a, a')} - \omega([a', a''], a) +$$

$$+ a.\omega(a', a'') - \overbrace{a'.\omega(a, a'')}^{= +a'.\omega(a'', a)} + a''.\omega(a, a') = 0$$

Now, *M* is an abelian subalgebra of $\breve{A}$ and it will be interesting to take *M* as the analytic representation given in section §7.

The formula (3.4) has a "Witt flavor", and the given decomposition suggests that $\not A^G$ seems an infinite dimensional version of a Borel subalgebra of a split Lie algebra. Since there is a classification of a vast family of infinite dimensional split Lie algebras ([Kac - Mathieu]), it will be very interesting, in order to understand the structure of our Lie algebra, to embed it in a split infinite dimensional Lie algebra, and we thought on generalized Witt algebras [K. Zhao] as a starting point. But, as pointed out by A. Pianzola, there is no natural embedding of a Lie algebra graded by the positive integers in a Z-graded one. A second try was to consider the algebra $\mathbb{A}_{\rho}^{G} \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}]$.

§4. THE ALGEBRA $\mathbb{M} = \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathbb{A}^{G}$

In the complex linear space $\mathbb{M} = \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathbb{A}^{G}$ there is a well defined (it is a standard fact) Lie product $[\,.\,,\,.\,]$ such that

$$[t^{m} \otimes X_{\breve{v}}, t^{l} \otimes X_{\breve{w}}] = t^{m+l} \otimes [X_{\breve{v}}, X_{\breve{w}}] \tag{4.1}$$

An equivalent versión (that we will adopt here, because the formula (4.1), extended to arbitrary elements of $\mathbb{M}$ involves infinite sums) is to view $\mathbb{M}$ as the algebra of the functions $x : S^{1} \times \mathbb{N} \to \mathbb{C}$ of the form

$$x(t,n) = \sum_{\breve{v} \in \Lambda} a_{\breve{v}}(t) X_{\breve{v}}(n) \tag{4.2}$$

where each $a_{\breve{v}}$ belongs to $\mathbb{C}[t,t^{-1}]$, i.e., it is a Laurent polynomial. Then, from (4.2) we have

$$x(t,n) = \sum_{\breve{v} \in \Lambda} a_{\breve{v}}(t) X_{\breve{v}}(n) = \sum_{\breve{v} \in \Lambda} \overbrace{\left(\sum_{m} a_{m\breve{v}} t^{m} \right)}^{\textit{finite sum}} X_{\breve{v}}(n) = \sum_{\breve{v}} \sum_{m} a_{m\breve{v}}\, t^{m} X_{\breve{v}}(n) \tag{4.3}$$

Denoting T^{m} the function such that $T^{m}(t) = t^{m}$, we have the simpler expression

$$x = \sum_{\breve{v}} \sum_{m} a_{m\breve{v}}\, T^{m} X_{\breve{v}} \tag{4.4}$$

Remark* 4.1*: The countable infinite system $\left\{T^m X_{\breve{v}} : m \in \mathbb{Z}, \breve{v} \in \Lambda\right\}$ is not a basis of $\mathcal{M}$ as a complex linear space, because the elements of $\mathcal{M}$ are <u>not</u> <u>finite</u> linear expressions of the elements of this system. What is true, is that for each n and each t, the last sum in (4.3) is finite. We will call this system as a *f.s.p.c.basis* ("finite sum pointwise convergence" basis).

Now, given another $y = \sum_{\breve{v}} \sum_{m} b_{m\breve{v}} T^m X_{\breve{v}}$:

$$[x, y] = \sum_{\breve{v},\breve{w}} \sum_{m,l} a_{m\breve{v}} b_{l\breve{w}} T^{m+l} [X_{\breve{v}}, X_{\breve{w}}] = \sum_{\breve{v},\breve{w}} \sum_{m,l} \sum_{\breve{u} \in \Lambda(\breve{v},\breve{w})} a_{m\breve{v}} b_{l\breve{w}} (\dot{w} - \dot{v}) \delta^{\dot{u}}_{\dot{v}+\dot{w}} T^{m+l} X_{\breve{u}} \tag{4.5}$$

Particularly:

$$[T^m X_{\breve{v}}, T^l X_{\breve{w}}] = T^{m+l} [X_{\breve{v}}, X_{\breve{w}}] = (\dot{w} - \dot{v}) T^{m+l} \sum_{\breve{u} \in \Lambda(\breve{v},\breve{w})} \delta^{\dot{u}}_{\dot{v}+\dot{w}} X_{\breve{u}} \tag{4.6}$$

(Compare with (4.1)). So, if we define degrees in the almost obvious way

$$Deg(T^m X_{\breve{v}}) = m + \dot{v} \tag{4.7}$$

we have, from (4.6), that $[T^m X_{\breve{v}}, T^l X_{\breve{w}}]$ is a finite sum of elements of the same degree $m + l + \dot{v} + \dot{w} = Deg(T^m X_{\breve{v}}) + Deg(T^l X_{\breve{w}})$.

Then, the degrees may take any integer value, positive or negative (or 0, of course). But there is a problem here for the classification of $\mathcal{M}$ as graduated algebra: given an integer k, there are infinitely many linearly independent basic elements $T^m X_{\breve{v}}$ of degree k (over $\mathbb{C}$ and over $\mathbb{C}[t,t^{-1}]$, if we view $\mathcal{M}$ as a module over $\mathbb{C}[t,t^{-1}]$). And without "finite growth", any try of classification is hopeless (although there are some Kac families…).

The next (third) try was to study another kind of relationship between $\mathcal{A}^G_\rho$ and the Witt algebra.

§5. AN EXACT SEQUENCE THAT MAY HELP

We first recall the formula $[X_{\breve{v}}, X_{\breve{w}}] = (\dot{w} - \dot{v}) X_{\breve{v}} * X_{\breve{w}} = (\dot{w} - \dot{v}) \sum_{\breve{u} \in \Lambda(\breve{v},\breve{w})} \delta^{\dot{u}}_{\dot{v}+\dot{w}} X_{\breve{u}}$ where $\Lambda(\breve{v},\breve{w}) = \{\breve{u} \in \Lambda : \max\{o(\breve{v}), o(\breve{w})\} \le o(\breve{u}) \le o(\breve{v}) + o(\breve{w})\}$. It follows that the linear subspaces

$$J_m = \bigoplus_{o(\breve{v}) \ge m} \mathbb{C} X_{\breve{v}} \tag{5.1}$$

are, in fact, ideals in $\mathcal{A}^G_\rho$. Now, for each positive integer *m*, let $(m) = (m,0,0,...) \in \Lambda$ (these are the elements of order 1). Observe that $\rho((m)) = (m)^\bullet = m$. For each couple of positive integers *m* y *n* we have $\Lambda((m),(n)) = \{\breve{u} \in \Lambda : 1 \le o(\breve{u}) \le 2\}$, so

$$[X_{(m)}, X_{(n)}] = (n-m) \sum_{1 \le o(\breve{u}) \le 2} \delta^{\breve{u}}_{m+n} X_{\breve{u}} = (n-m)\{X_{(m+n)} + X_{(m,n,0,0...)}\} \tag{5.2}$$

(we assumed that $m > n$, otherwise $(n,m,0,0,\ldots)$ must replace $(m,n,0,0,\ldots)$; if $m = n$ everything vanishes and the universe tends to disappear….). Then if we consider the subspace *S* spaned by the elements of order 1 (or 0…), i.e.,

$$S = \bigoplus_{m=0}^{\infty} \mathbb{C} X_{(m)} \tag{5.3}$$

(it is not an ideal, nor a subalgebra), then:

(*i*) $\mathcal{A}^G_\rho = (\bigoplus_{m=1}^{\infty} \mathbb{C} X_{(m)}) \oplus (\bigoplus_{o(\breve{v}) \ge 2} \mathbb{C} X_{\breve{v}}) = S \oplus J_2$ (these sums must be understood in the "finite pointwise convergence" sense)

(*ii*) In the quotient $\mathcal{A}^G_\rho / J_2$, we have $[X_{(m)} + J_2, X_{(n)} + J_2 \tilde{]} = (n-m) X_{(m+n)} + J_2$, that it to say:

$$[\tilde{X}_{(m)}, \tilde{X}_{(n)} \tilde{]} = (n-m) \tilde{X}_{(m+n)} \tag{5.4}$$

This suggests the following "half" of the Witt algebra: in the algebra $\mathbb{C}[z]$ let us define, for each nonnegative integer *m*, the derivation $D_m = -z^{m+1}\dfrac{d}{dz}$. Then,

$$[D_m, D_n] \overset{def}{=} D_m \circ D_n - D_n \circ D_m = (n-m)D_{m+n}\,. \qquad (5.5)$$

Observe that $\mathcal{L} = \underset{m\geq 0}{\oplus} \mathbb{C}D_m$ (we were tempted to denote $\mathcal{W}^{+}$ this algebra) is a $\mathbb{Z}_{\geq 0}$-graded Lie algebra, where $[D_0, D_n] = nD_n$, that is to say: each homogeneous term $\mathcal{L}_m = \mathbb{C}D_m$ is an eigenspace of D_0 corresponding to the eigenvalue *m*. Now, we have the following diagram of Lie algebra morphisms:

$$\begin{array}{ccccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \mathcal{L} & & \\ & & & & \downarrow \varphi & & \\ 0 \longrightarrow & J_2 & \longrightarrow & \mathcal{A}_\rho^G & \overset{\pi}{\longrightarrow} & \mathcal{A}_\rho^G / J_2 & \longrightarrow 0 \end{array} \qquad (5.6)$$

where φ is the (injective) Lie algebra morphism such that $\varphi(D_n) = \tilde{X}_{(n)}$. Given that we are working over a field (= $\mathbb{C}$; so all projective modules are free, particularly $\mathcal{L}$ is projective: this leads to a natural generalization), there exists a linear mapping $h : \mathcal{A}_\rho^G \to \mathcal{L}$ such that $\pi = \varphi \circ h$. Now, given that π and φ are Lie algebra morphisms, for each couple of elements a and b in $\mathcal{A}_\rho^G$:

$$0 = \pi([a,b]) - [\pi(a), \pi(b)\tilde{]} = \varphi(h([a,b])) - [\varphi(h(a)), \varphi(h(b))\tilde{]} =$$
$$= \varphi(h([a,b])) - \varphi([h(a), h(b)]_{\mathcal{L}}) = \varphi(h([a,b]) - [h(a), h(b)]_{\mathcal{L}})$$

But φ is injective, so $h = A_\rho^G \to Ł$ **is a Lie algebra morphism**; and for the same reason (the injectivity of φ), $Ker\ (h) = Ker\ (\pi) = J_2$. On the other hand:

$$0 = \widetilde{X}_{(m)} - \pi(X_{(m)}) = \varphi(D_m) - \pi(X_{(m)}) = \varphi(D_m) - \varphi(h(X_{(m)}) = \varphi(D_m - h(X_{(m)}))$$

Again, the injectivity of φ implies that $h(X_{(m)}) = D_m$. But these elements span $Ł$ as a linear space, so h is surjective. That is to say: we have an exact sequence of Lie algebra morphisms

$$0 \longrightarrow J_2 \longrightarrow A_\rho^G \xrightarrow{h} Ł \longrightarrow 0 \qquad (5.7)$$

Now, observe that A_ρ^G and J_2 are $Ł$ - modules by the almost obvious formula

$$D_m . X_{\check{v}} \overset{def}{=} [X_{(m)}, X_{\check{v}}] \qquad (5.8)$$

(it's checked). Then, the short exact sequence (5.7) gives rise to the long exact sequences

$$\ldots \to H^i(Ł, J_2) \to H^i(Ł, A_\rho^G) \to H^i(Ł, Ł) \to H^{i+1}(Ł, J_2) \to H^{i+1}(Ł, A_\rho^G) \to H^{i+1}(Ł, Ł) \to \ldots$$

$$(5.9)$$

$$\ldots \to H_i(Ł, J_2) \to H_i(Ł, A_\rho^G) \to H_i(Ł, Ł) \to H_{i-1}(Ł, J_2) \to H_{i-1}(Ł, A_\rho^G) \to H_{i-1}(Ł, Ł) \to \ldots$$

It is a standard fact (cf. [11], for instance). The cohomology referred to may be defined (there are several equivalent ways) by the Chevalley-Eilenberg complex and we have the following program:

1) Understand the details in the first long exact sequence

2) Compute the cohomology of $Ł$ and try to obtain some information about the cohomology of our Lie algebra A_ρ^G

3) $\mathcal{A}_\rho^G \cong J_2 \oplus \mathcal{L}$ is a direct sum of linear spaces. But what about the Lie bracket? There is a cocycle around: $\mathcal{A}_\rho^G$ is a (non central) extension of $\mathcal{L}$.

4) Try to fit the Witt algebra in the sequence (5.7) (recall that $\mathcal{L}$ is "W^+")

5) Study twisted forms of $\mathcal{A}_\rho^G \otimes \mathbb{C}[t,t^{-1}]$

6) Try to find a Killing form in $\mathcal{A}^G$. In the associative algebra $\mathcal{A}^G$ we have the invariant $\mathbb{C}$ - bilinear symmetric form

$$B(a,b) = \sum_{n=1}^{\infty} \frac{(a*b)(n)}{n!} \tag{5.10}$$

Actually, this is an invariant symmetric bilinear form in the hole associative algebra $\mathcal{A}$, with the corresponding restriction imposed by convergence: we may consider, for instance, just the arithmetic functions of polynomial order. It is obvious that $B(a*b,c) = B(a,b*c)$. What we could not find is an invariant form in the Lie algebra $\mathcal{A}_\rho^G$.

7) The group S_∞ is not the direct limit of the symmetric groups S_n. The direct limit of these groups is the subgroup $S(\infty)$, the permutations of finite support. The representations of $S(\infty)$ are studied in recent works (v. gr. [9]). For the topology introduced in S_∞, the subgroup $S(\infty)$ is dense on the full group. It would be interesting to have some result on the classifications of symmetric functions in this context. For instance, we have the invariant functions

$$\rho(n) = \sum_{k=1}^{\infty} v_{p_k}(n),\ \rho_2(n) = \sum_{1 \le k < l} v_{p_k}(n) v_{p_l}(n)\ ,\ldots,\ \rho_h(n) = \sum_{1 \le k_1 < k_2 < \ldots < k_h} v_{p_{k_1}}(n) v_{p_{k_2}}(n) \ldots v_{p_{k_h}}(n)\ ,\ \ldots$$

It seems plausible that these functions generate in some sense the full algebra of invariant arithmetic functions. This idea is reinforced at the analytic representation level, because the generalized prime zeta functions may be obtained (Newton' formulae) as polynomials in $P(k_1(\breve{v})s), P(k_2(\breve{v})s), P(k_3(\breve{v})s), \ldots$ where $P(s)$ is the prime zeta function and the $k_i(\breve{v})$ are the elementary symmetric functions of $v_1, v_2, \ldots, v_r$ (see next section)

§6. ANALYTIC REPRESENTATION OF THE ASSOCIATIVE ALGEBRA $\mathcal{A}^G$

Following the idea of Shapiro [HNS] with respect to analytic representations of the ring of arithmetic functions, let us define, for each $a \in \mathcal{A}$, the formal Dirichlet-power series in two variables

$$\Phi(a)(s,z) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} z^{\rho(n)} \tag{6.1}$$

Observe that for $z = 1$ this formula gives the classical Dirichlet series of a. Let us denote $\Im$ the complex algebra of these formal series. It is easy to see that $\Phi : \mathcal{A} \to \Im$ is $\mathbb{C}$-algebra morphism: the multiplicative property

$$\Phi(a * b) = \Phi(a)\Phi(b) \tag{6.2}$$

is due essentially to the additive property of ρ :

$$(\Phi(a)\Phi(b))(s,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a(m)b(n)z^{\rho(m)}z^{\rho(n)}}{m^s n^s} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a(m)b(n)z^{\rho(m)+\rho(n)}}{(mn)^s} =$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a(m)b(n)z^{\rho(mn)}}{(mn)^s} = \sum_{k=1}^{\infty}\left(\sum_{mn=k} a(m)b(n)\right)\frac{z^{\rho(k)}}{k^s} = \sum_{k=1}^{\infty}(a*b)(k)\frac{z^{\rho(k)}}{k^s} = \Phi(a*b)(s,z)$$

By the way: this is the basic fact in the idea of analytic representations in [HNS]. Now, recall that the choice of the logarithmic function ρ is motivated by the study of the representations of the invariant arithmetic functions. Given $a \in \mathcal{A}_\rho^G$:

$$\Phi(a)(s,z) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} z^{\rho(n)} = \sum_{\breve{v} \in \Lambda} a(n_{\breve{v}}) z^{\dot{v}} \overbrace{\left(\sum_{n \in G.n_{\breve{v}}} \frac{1}{n^s}\right)}^{P_{\breve{v}}(s)} = \sum_{\breve{v} \in \Lambda} a(n_{\breve{v}}) P_{\breve{v}}(s) z^{\dot{v}} \tag{6.3}$$

where the functions

$$P_{\breve{v}}(s) = \sum_{n \in G.n_{\breve{v}}} \frac{1}{n^s} = \sum_{\underbrace{\text{distitinct primes}}_{q_1, q_2, \ldots, q_r}} \frac{1}{(q_1^{v_1} q_2^{v_2} \ldots q_r^{v_r})^s} \tag{6.4}$$

may be called “generalized prime zeta functions”, for obvious reasons.

***Some important examples and some properties of the generalized prime zeta functions*:**

(1) $\Phi(e_1)(s,z)=\sum_{n=1}^{\infty}\frac{e_1(n)}{n^s}z^{\rho(n)}=\frac{1}{1^s}z^{\rho(1)}=z^0=1$

(2) $\Phi(X_{\breve{v}})(s,z)=\sum_{n=1}^{\infty}\frac{\chi_{\breve{v}}(n)}{n^s}z^{\rho(n)}=\sum_{n\in G.n_{\breve{v}}}\frac{1}{n^s}z^{\dot{v}}=z^{\dot{v}}\sum_{n\in G.n_{\breve{v}}}\frac{1}{n^s}=z^{\dot{v}}P_{\breve{v}}(s)$.

(3) $\Phi(1)(s,z)=\sum_{n=1}^{\infty}\frac{1}{n^s}z^{\rho(n)}=\sum_{\breve{v}\in\Lambda}\sum_{n\in G.n_{\breve{v}}}\frac{1}{n^s}z^{\dot{v}}=\sum_{\breve{v}\in\Lambda}z^{\dot{v}}\sum_{n\in G.n_{\breve{v}}}\frac{1}{n^s}=\sum_{\breve{v}\in\Lambda}z^{\dot{v}}P_{\breve{v}}(s)$. Observe that for z = 1 we have $\Phi(1)(s,1)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\varsigma(s)$, then:

$$\sum_{\breve{v}\in\Lambda}P_{\breve{v}}(s)=\varsigma(s) \qquad (6.5)$$

Of course, this identity may be obtained directly from the definitions: the sum in the left is a sum $\sum\frac{1}{n^s}$ over the set of positive integers partitioned in the orbits of *G*.

(4) $\Phi(\mu)(s,z)=\sum_{\breve{v}\in\Lambda}\mu(n_{\breve{v}})P_{\breve{v}}(s)z^{\dot{v}}$. But $\mu(n_{\breve{v}})=\mu(p_1^{v_1}...p_r^{v_r})=0$ if some $v_i\geq 2$ and $\mu(p_1p_2...p_r)=(-1)^r$. Then, $\Phi(\mu)(s,z)=\sum_{r=0}^{\infty}(-1)^r P_r(s)z^r$, where

$$P_r(s)=P_{\underbrace{(1,1,...,1)}_{r}}(s)=\sum_{\underbrace{q_1,q_2,...,q_r}_{disticnt\ \ primes}}\frac{1}{(q_1q_2...q_r)^s} \qquad (6.6)$$

Again, taking z = 1 we obtain

$$\frac{1}{\varsigma(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=\Phi(\mu)(s,1)=\sum_{r=0}^{\infty}(-1)^r P_r(s) \qquad (6.7)$$

This identity is less trivial than (6.5), but may be obtained directly, too.

(5) $\Phi(\rho)(s,z) \overset{(4.3)}{=} \sum_{\breve{v}\in\Lambda} \rho(n_{\breve{v}}) P_{\breve{v}}(s) z^{\dot{v}} = \sum_{\breve{v}\in\Lambda} \dot{v} P_{\breve{v}}(s) z^{\dot{v}} = z\frac{\partial}{\partial z}\left(\sum_{\breve{v}\in\Lambda} P_{\breve{v}}(s) z^{\dot{v}}\right) = z\frac{\partial}{\partial z}\Phi(1)(s,z)$

(6) Recall that the Liouville function is $\lambda(n) = (-1)^{\rho(n)}$, then:

$$\Phi(\lambda)(s,z) \overset{(4.3)}{=} \sum_{\breve{v}\in\Lambda} \lambda(n_{\breve{v}}) P_{\breve{v}}(s) z^{\dot{v}} = \sum_{\breve{v}\in\Lambda} (-1)^{\rho(n_{\breve{v}})} P_{\breve{v}}(s) z^{\dot{v}} = \sum_{\breve{v}\in\Lambda} (-1)^{\dot{v}} P_{\breve{v}}(s) z^{\dot{v}}$$

$$= \sum_{\breve{v}\in\Lambda} P_{\breve{v}}(s)(-z)^{\dot{v}} = \Phi(1)(s,-z)$$

On the other hand, for $z = 1$: $\frac{\varsigma(2s)}{\varsigma(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \Phi(\lambda)(s,1) = \sum_{\breve{v}\in\Lambda} (-1)^{\dot{v}} P_{\breve{v}}(s)$ (6.8)

To close this section, we observe that the multiplication formula for the generalized prime zeta function is

$$\begin{aligned} P_{\breve{v}}(s)P_{\breve{w}}(s) &= \Phi(X_{\breve{v}})(s,1)\Phi(X_{\breve{w}})(s,1) = \Phi(X_{\breve{v}} * X_{\breve{w}})(s,1) = \\ &= \Phi\left(\sum_{\breve{u}\in\Lambda(\breve{v},\breve{w})} \delta^{\dot{u}}_{\dot{v}+\dot{w}} X_{\breve{u}}\right)(s,1) = \sum_{\breve{u}\in\Lambda(\breve{v},\breve{w})} \delta^{\dot{u}}_{\dot{v}+\dot{w}} \Phi(X_{\breve{u}})(s,1) = \sum_{\breve{u}\in\Lambda(\breve{v},\breve{w})} \delta^{\dot{u}}_{\dot{v}+\dot{w}} P_{\breve{u}}(s) \end{aligned} \tag{6.9}$$

where (recall) $\Lambda(\breve{v},\breve{w}) = \{\breve{u}\in\Lambda : \max\{o(\breve{v}), o(\breve{w})\} \le o(\breve{u}) \le o(\breve{v}) + o(\breve{w})\}$. For instance, a very simple computation shows that $P_{(1)}(s)P_{(2)}(s) = P_{(2,1)}(s) + P_{(3)}(s)$, where $(m) = (m,0,0,...)$ and $(2,1) = (2,1,0,0,0,...)$. (Observe that this is not the formula for the Schur polynomials).

PROPOSITION 6.1. The mapping $\Phi : \mathcal{A}^G \to \mathcal{D}$ is injective.[1]

***Proof*:**

(a) For each positive integer *n*, let $E_n : \mathbb{C} \to \mathbb{C}$ the function $s \mapsto n^{-s} = e^{-s\log(n)}$. Then, the set $\{E_n : n\in\mathbb{N}\}$ is linearly independent over $\mathbb{C}$: *Uniqueness Theorem* for Dirichlet series.

(b) In the complex linear space spanned by $\{E_n : n \in \mathbb{N}\}$, define the scalar product $< . , . >$ such that $<E_n, E_m> = \frac{\delta_{nm}}{\sqrt{n!m!}}$ (item (a) gives consistency of this definition). Now, extend this scalar product to the space of Dirichlet series $\sum_{n=1}^{\infty} a(n)E_n$ with some fixed abscissa of absolute convergence. That is to say: $\left\langle \sum_{n=1}^{\infty} a(n)E_n \, , \sum_{m=1}^{\infty} b(m)E_m \right\rangle = \sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n!}$.

(c) $\breve{v} \neq \breve{w} \Rightarrow \langle P_{\breve{v}}, P_{\breve{w}} \rangle = 0$ and $\|P_{\breve{v}}\|^2 = \sum_{n \in G.n_{\breve{v}}} \frac{1}{n!}$:

$$\langle P_{\breve{v}}, P_{\breve{w}} \rangle = \left\langle \sum_{n=1}^{\infty} X_{\breve{v}}(n)E_n \, , \sum_{m=1}^{\infty} X_{\breve{w}}(m)E_m \right\rangle = \sum_{n=1}^{\infty} \frac{X_{\breve{v}}(n)\overline{X_{\breve{w}}(n)}}{n!} .$$

Now, $X_{\breve{v}}(n)\overline{X_{\breve{w}}(n)} = X_{\breve{v}}(n)X_{\breve{w}}(n) \neq 0$ iff $n \in G.n_{\breve{v}} \cap G.n_{\breve{w}} \overset{\breve{v} \neq \breve{w}}{=} \phi$.

(d) The set $\{P_{\breve{v}} : \breve{v} \in \Lambda\}$ is linearly independent (consequence of (c)), so (6.3) implies that $Ker\ \Phi = \{0\}$: $\Phi(a) = 0 \Rightarrow \sum_{\breve{v} \in \Lambda} a(n_{\breve{v}})P_{\breve{v}} = 0 \Rightarrow \forall \breve{v} \in \Lambda : a(n_{\breve{v}}) = 0 \overset{a \in A^G}{\Rightarrow} a = 0$ □

[1] ***Remark* 6.1**: The algebra Ɖ is, at the present, the algebra of formal "Dirichlet (*s*-*z*)-series". In the proof we used non formal series and this point has to be clarified. We may restrict the ring to invariant arithmetic functions of polynomial order, or to appeal to the uniqueness theorem for Dirichlet series to resume the proof. But the scalar product given in the proof (it works for the polynomial order arithmetic functions) seems to be interesting in itself.

§7. ANALYTIC REPRESENTATION OF THE LIE ALGEBRA $\mathcal{A}_\rho^G$

With the same notations of the preceding section, we have

$$\Phi([a,b]) = \Phi(a * \rho b - \rho a * b) = \Phi(a)\Phi(\rho b) - \Phi(\rho a)\Phi(b) \ . \qquad (7.1)$$

On the other hand,

$$\Phi(\rho a)(s,z)=\sum_{n=1}^{\infty}\frac{\rho(n)a(n)}{n^s}z^{\rho(n)}=z\frac{\partial}{\partial z}\sum_{n=1}^{\infty}\frac{a(n)}{n^s}z^{\rho(n)}=z\frac{\partial}{\partial z}\Phi(a)(s,z) \quad (7.2)$$

Then:

$$\Phi([a,b])=\Phi(a)\Phi(\rho b)-\Phi(\rho a)\Phi(b)=\Phi(a)\left\{z\frac{\partial}{\partial z}\Phi(b)\right\}-\left\{z\frac{\partial}{\partial z}\Phi(a)\right\}\Phi(b)$$

Now, it is clear that $\Phi : \mathcal{A}_{\rho}^{G} \to \mathfrak{I}$ is a morphism of complex Lie algebras, where the bracket in $\mathfrak{I}$ is

$$[\alpha,\beta]=\alpha\ z\frac{\partial\beta}{\partial z}-z\frac{\partial\alpha}{\partial z}\ \beta \quad (7.3)$$

With respect to the basis $\{X_{\breve{v}} : \breve{v}\in\Lambda\}$ we have (recall)

$$[X_{\breve{v}},X_{\breve{w}}]=(\dot{w}-\dot{v})X_{\breve{v}} * X_{\breve{w}}=(\dot{w}-\dot{v})\sum_{\breve{u}\in\Lambda(\breve{v},\breve{w})}\delta_{\dot{v}+\dot{w}}^{\dot{u}}X_{\breve{u}}$$

and the corresponding analytic version is

$$[z^{\dot{v}}P_{\breve{v}},z^{\dot{w}}P_{\breve{w}}]=(\dot{w}-\dot{v})z^{\dot{v}+\dot{w}}P_{\breve{v}}P_{\breve{w}}=z^{\dot{v}+\dot{w}}(\dot{w}-\dot{v})\sum_{\breve{u}\in\Lambda(\breve{v},\breve{w})}\delta_{\dot{v}+\dot{w}}^{\dot{u}}P_{\breve{u}}$$

§8. THE ANALYTIC EXTENSION PROBLEM

In order to make the analytic representation a useful interaction between the complex analysis and the algebraic structure of the algebras $\mathcal{A}^{G}$ and $\mathcal{A}_{\rho}^{G}$, we are confronted with a difficult problem: the analytic extension of the generalized prime zeta functions. There is a vast literature about this kind of problem, starting with the making epoch papers of Riemann. We are not too ambitious here. What we want is to find a “good” domain for a meromorphic extension of the generalized prime zeta functions. After that, with restriction of the invariant arithmetic functions to the polynomial order, we hope that this domain will stand for the representation of these functions.

We first observe that

1) For each $\breve{v}=(v_1,v_2,...,v_r,0,0,...)$: $\sum_{n\in G.n_{\breve{v}}}\frac{1}{|n^s|}=\sum_{n\in G.n_{\breve{v}}}\frac{1}{n^\sigma}\leq\sum_{n=1}^{\infty}\frac{1}{n^\sigma}$, so that $P_{\breve{v}}(s)=\sum_{n\in G.n_{\breve{v}}}\frac{1}{n^s}$ is holomorphic at least in the half plane $\mathbb{H}_1=\{\sigma+it\in\mathbb{C}:\sigma>1\}$.

2) The identity (6.7): $\frac{1}{\varsigma(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=\Phi(\mu)(s,1)=\sum_{r=0}^{\infty}(-1)^r P_r(s)$, where $P_r=P_{\underbrace{(1,1,...,1,0,0,...0}_{r}}$, imposes a frontline for the possible extensions: the critical line of nontrivial zeroes of the Riemann *z*-function. On the other hand, the identity $\sum_{\breve{v}\in\Lambda}P_{\breve{v}}(s)=\varsigma(s)$ seems to indicate that the singularities of the generalized prime zeta functions may cancel in $\mathbb{H}_0=\{\sigma+it\in\mathbb{C}:\sigma>0\}$ (and beyond this half plane, it is well known that there is nothing to do with the prime zeta functions)

PROPOSITION 8.1: Let $\breve{v}=(v_1,v_2,...,v_r,0,0,...)$, $v_1\geq v_2\geq...\geq v_2\geq 1$ and the corresponding $P_{\breve{v}}(s)=\sum_{n\in G.n_{\breve{v}}}\frac{1}{n^s}=\sum_{\substack{\text{distitinct primes}\\ q_1,q_2,...,q_r}}\frac{1}{(q_1^{v_1}q_2^{v_2}...q_r^{v_r})^s}$. Then this series converges absolutely in the half plane $\mathbb{H}_{\frac{1}{v_r}}=\left\{s=\sigma+it\in\mathbb{C}:\sigma>\frac{1}{v_r}\right\}$.

Proof: The absolute value of each $n^s=n^{\sigma+it}\overset{def}{=}n^\sigma e^{it\log(n)}$ is n^σ. For each set $\{q_1,q_2,...,q_r\}$ of distinct primes, we have $q_1^{v_1}q_2^{v_2}...q_r^{v_r}\geq q_1^{v_r}q_2^{v_r}...q_r^{v_r}=(q_1q_2...q_r)^{v_r}$. Let us denote $\mathbb{P}(r)$ the family of subsets $\{q_1,q_2,...,q_r\}\subset\mathbb{P}$ of *r* distinct primes. Observe that the ordering of $\mathbb{P}(r)$ by set inclusion is a mess, but there is a simpler and net ordering that is very well suited for our purposes: $\{q_1,q_2,...,q_r\}\prec\{q_1',q_2',...,q_r'\}\Leftrightarrow q_1q_2...q_r<q_1'q_2'...q_r'$. The partial sums of the series $\sum_{\substack{\text{distitinct primes}\\ q_1,q_2,...,q_r}}\frac{1}{(q_1^{v_1}q_2^{v_2}...q_r^{v_r})^s}$ will be understood in this sense, that is to say:

$$\sum_{\substack{\text{distitinct primes}\\ q_1,q_2,...,q_r}}\frac{1}{(q_1^{v_1}q_2^{v_2}...q_r^{v_r})^s}=\lim_{m\to\infty}\sum_{\substack{\{q_1,q_2,...,q_r\}\in P(r)\\ q_1q_2...q_r\leq m}}\frac{1}{(q_1^{v_1}q_2^{v_2}...q_r^{v_r})^s}\qquad(8.1)$$

Then, given $\sigma>0$, for each positive integer *m*:

$$\sum_{\substack{\{q_1,q_2,...,q_r\}\in P(r)\\ q_1q_2...q_r\le m}}\frac{1}{(q_1^{v_1}q_2^{v_2}...q_r^{v_r})^{\sigma}}\le\sum_{\substack{\{q_1,...,q_r\}\in P(r)\\ q_1q_2...q_r\le m}}\frac{1}{(q_1q_2...q_r)^{v_r\sigma}}=$$

$$=\sum_{\substack{\{q_1,...q_r\}\in P(r)\\ q_1q_2...q_r\le m\\ q_1<q_2<...<q_r}}\sum_{\gamma\in S_r}\frac{1}{(q_{\gamma(1)}q_{\gamma(2)}...q_{\gamma(r)})^{v_r\sigma}}=\sum_{\substack{\{q_1,...q_r\}\in P(r)\\ q_1q_2...q_r\le m\\ q_1<q_2<...<q_r}}\frac{r!}{(q_1q_2...q_r)^{v_r\sigma}}\le r!\sum_{n=1}^{\infty}\frac{1}{n^{v_r\sigma}} \qquad (8.2)$$

The last series converges for $v_r\sigma > 1$ □

COROLLARY 8.2 (of the proof). For each $\sigma > \frac{1}{v_r}$: $P_{\bar{v}}(\sigma)\le r!\varsigma(\sigma v_r)$ (The last series in (8.2) is $r!\varsigma(\sigma v_r)$ for $\sigma v_r > 1$)□

Remark* 8.1*: For each $\varepsilon > 0$ the series (8.1) converges absolutely and uniformly in the closed half plane $\overline{\mathbb{H}}_{\frac{1}{v_r}+\varepsilon}=\left\{ s=\sigma+it\in\mathbb{C}:\sigma\ge\frac{1}{v_r}+\varepsilon\right\}$ (Weierstrass criterion). Then, $P_{\bar{v}}$ is holomorphic in $\mathbb{H}_{\frac{1}{v_r}}=\left\{ s=\sigma+it\in\mathbb{C}:\sigma>\frac{1}{v_r}\right\}$ (Morera Theorem, etc.).

***Remark* 8.2**: There are some coarse inequalities in (8.2). Perhaps a better result may be obtained with more subtle work. The role of the number v_r seems to be exaggerated, taking account that it is not an invariant. On the other hand, from the inequality satisfied by the geometric and arithmetic averages, in our case: $(q_1^{v_1}q_2^{v_2}...q_r^{v_r})^{\frac{1}{r}}\le\frac{1}{r}\left(q_1^{v_1}+q_2^{v_2}+...+q_r^{v_r}\right)$, that is to say $\frac{1}{q_1^{v_1}q_2^{v_2}...q_r^{v_r}}\ge\frac{r^r}{(q_1^{v_1}+q_2^{v_2}+...+q_r^{v_r})^r}$. Then:

$$\sum_{\{q_1,q_2,...,q_r\}\in P(r)}\frac{1}{(q_1^{v_1}q_2^{v_2}...q_r^{v_r})^{\sigma}}\ge\sum_{\{q_1,q_2,...,q_r\}\in P(r)}\frac{r^{r\sigma}}{(q_1^{v_1}+q_2^{v_2}+...+q_r^{v_r})^{r\sigma}}\ge$$

$$=r^{r\sigma}\sum_{\substack{\{q_1,q_2,...,q_r\}\in P(r)\\ q_1<q_2<...<q_r}}\sum_{\gamma\in S_r}\frac{1}{(q_{\gamma(1)}^{v_1}+q_{\gamma(2)}^{v_2}+...+q_{\gamma(r)}^{v_r})^{r\sigma}}\ge$$

$$\geq r^{r\sigma} \sum_{\substack{\{q_1,q_2,\ldots,q_r\}\in P(r)\\ q_1<q_2<\ldots<q_r}} \sum_{\gamma\in S_r} \frac{1}{(q_{\gamma(1)}^{v_1}+q_{\gamma(2)}^{v_1}+\ldots+q_{\gamma(r)}^{v_1})^{r\sigma}} =$$

$$= r^{r\sigma} r! \sum_{\substack{\{q_1,q_2,\ldots,q_r\}\in P(r)\\ q_1<q_2<\ldots<q_r}} \frac{1}{(q_1^{v_1}+q_2^{v_1}+\ldots+q_r^{v_1})^{r\sigma}} \geq r^{r\sigma} r! \sum_{\substack{\{q_1,q_2,\ldots,q_r\}\in P(r)\\ q_1<q_2<\ldots<q_r}} \frac{1}{(q_r^{v_1}+q_r^{v_1}+\ldots+q_r^{v_1})^{r\sigma}} =$$

$$= r^{r\sigma} r! \sum_{\substack{\{q_1,q_2,\ldots,q_r\}\in P(r)\\ q_1<q_2<\ldots<q_r}} \frac{1}{q_r^{v_1 r\sigma}} \geq r^{r\sigma} r! \sum_{\substack{p\in P\\ p_{r+1}\leq p}} \frac{1}{p^{v_1 r\sigma}} = -r^{r\sigma} r! \sum_{\substack{p\in P\\ p_1\leq p\leq p_r}} \frac{1}{p^{v_1 r\sigma}} + r^{r\sigma} r! \sum_{p\in P} \frac{1}{p^{v_1 r\sigma}}$$

About the last inequality: for each prime $p \geq p_{r+1}$ (= the (r+1)-th prime in the natural sequence of positive primes), there is at least one sequence $q_1 < q_2 < \ldots < q_r$ of primes such that $q_r = p$

This is known since 1737 (Euler, who else?), that $\sum_{p\in P}\frac{1}{p} = \infty$. Then, if $v_1 r\sigma \leq 1$, the series $\sum_{\{q_1,q_2,\ldots,q_r\}\in P(r)} \frac{1}{(q_1^{v_1}q_2^{v_2}\ldots q_r^{v_r})^{\sigma}}$ diverges. It follows that the abscissa σ_a of absolute convergence for the Dirichlet series $\sum_{\{q_1,q_2,\ldots,q_r\}\in P(r)} \frac{1}{(q_1^{v_1}q_2^{v_2}\ldots q_r^{v_r})^{\sigma+it}}$ verifies

$$\frac{1}{rv_1} < \sigma_a \leq \frac{1}{v_r} \tag{8.3}$$

(About the abscissas of convergence of Dirichlet series, see [2], for instance; specially suited for our purposes seems to be the theorem of Landau on this subject).

***Remark* 8.3**: For the algebra $\mathcal{A}_\rho^G \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}]$, the analytic representation should be $\Phi(a)(s,t) = \sum_{\breve{v},m} a_{\breve{v}m} P_{\breve{v}}(s) t^{m+\dot{v}}$?. Observe that the formula $a_{\breve{v}m} = a(n_{\breve{v}})\delta_m^0$ gives a natural immersion $\mathcal{A}_\rho^G \to \mathcal{A}_\rho^G \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}]$.

§9. FINAL DISPERSE NOTES AND APENDICES

§9.1. An interesting invariant arithmetic function

When we tried to study the Lie subalgebra of $\mathcal{A}_{\rho}^{G}$ with generators $1, \mu$, we find that the characteristic function J of the set $\{p^m : p \in \mathbb{P}, m \in \mathbb{N}\}$ verifies: $[1, \mu] = -2J$. We regret the lack of time to pursue this line. We give the computation here.

Recall: $[a,b] = a * (\rho b) - (\rho a) * b$. Then:

(1) For each b: $[e_1, b] = \rho b$.

P/ $[e_1, b] = e_1 * (\rho b) - (\rho e_1) * b = \rho b$, because $\rho e_1 = 0$ □

(2) For each a and b: $\rho(a*b) + [a,b] = 2(a * \rho b)$ y $\rho(a*b) - [a,b] = 2(\rho a * b)$

P/ $\rho(a*b) = (\rho a) * b + a * (\rho b)$ etc.□

(3) Let $J : \mathbb{N} \to \{0,1\}$ be the characteristic function of the set $\{p^m : p \in \mathbb{P}, m \in \mathbb{N}\}$ (positive integer powers of primes). This function is clearly invariant). Then: $[1, \mu] = -2J$.

P/ $[1,\mu] \overset{(2)}{=} 2(1*(\rho\mu)) - \rho(1*\mu) = 2(1*(\rho\mu)) - \overbrace{\rho e_1}^{=0} = 2(1*(\rho\mu))$. So that we have to compute $(1*(\rho\mu))(n) = \sum_{d|n} \rho(d)\mu(d)$. Given $n = q_1^{\nu_1} q_2^{\nu_2} .. q_r^{\nu_r}$ (with $r \geq 2$ prime distinct factors q_1, $q_2, ..., q_r$), the non vanishing terms in $\sum_{d|n} \rho(d)\mu(d)$ are those for which the divisor d of n is different from 1 (because $\rho(1) = 0$) and does not contain quadratic divisors (because in this case $\mu(d) = 0$). The set of divisors of $n = q_1^{\nu_1} q_2^{\nu_2} .. q_r^{\nu_r}$ different from 1 and without quadratic factors is the disjoint union

$$\{q_i : 1 \leq i \leq r\} \cup \{q_i q_j : 1 \leq i, j \leq r\} \cup \{q_i q_j q_j : 1 \leq i < j < k \leq r\} \cup ... \cup \{q_1 q_2 q_3 .. q_r\}$$

Observe that the cardinals of this sets are, respectively, $r = \binom{r}{1}$, $\binom{r}{2}$, $\binom{r}{2}$, … ., $\binom{r}{r} = 1$, and $\rho\mu$ is constant of each of them. More precisely:

$\rho(q_i)\mu(q_i) = -1$,

$\rho(q_i q_j)\mu(q_i q_j) = 2$,

$\rho(q_i q_j q_k)\mu(q_i q_j q_k) = -3, \ldots$

$\rho(q_1 q_2 .. q_r)\mu(q_1 q_2 .. q_r) = r(-1)^r$

(recall that all the prime factors are different). Then:

$$(1*(\rho\mu))(q_1^{v_1}q_2^{v_2}...q_r^{v_r}) = \sum_{k=1}^{r}\binom{r}{k}k(-1)^k = -\binom{r}{1}+\binom{r}{2}2-\binom{r}{3}3+...+\binom{r}{r}r(-1)^r$$

On the other hand (here the assumption $r \geq 2$ is crucial), differentiating the identity $(1-x)^r = \sum_{k=0}^{r}\binom{r}{k}(-1)^k x^k = 1-\binom{r}{1}x+\binom{r}{2}x^2+...\binom{r}{r}(-1)^r x^r$ we obtain

$$-r(1-x)^{r-1} = \sum_{k=1}^{r}\binom{r}{k}k(-1)^k x^{k-1} = -\binom{r}{1}+\binom{r}{2}2x+...\binom{r}{r}r(-1)^r x^{r-1}$$

Evaluating at x =1:

$$0 = -r(1-1)^{r-1} = \sum_{k=1}^{r}\binom{r}{k}k(-1)^k = -\binom{r}{1}+\binom{r}{2}2+...\binom{r}{r}r(-1)^r$$

That is to say: if si $r \geq 2$: $(1*(\rho\mu))(q_1^{v_1}q_2^{v_2}...q_r^{v_r}) = 0$. Let us see the remaining cases: if n = 1 (this is the case r = 0, because r is the number of distinct prime factors of n), we obtain rivially $(1*(\rho\mu))(1) = \overbrace{\rho(1)}^{=0}\mu(1) = 0$. Finally, if r =1, n is of the form $n = p^m$ for some prime p and some positive integer m. Then,

$$(1*(\rho\mu))(p^m) = \sum_{d|p^m}\rho(d)\mu(d) = \overbrace{\rho(1)}^{=0}\mu(1)+\rho(p)\mu(p)+\rho(p^2)\overbrace{\mu(p^2)}^{=0}+...+\rho(p^m)\overbrace{\mu(p^m)}^{=0}$$

that is to say: $(1*(\rho\mu))(p^m) = \rho(p)\mu(p) = -1$. □

§9.2. A remark on *p*-adic valuations and Bell series

Given a ring morphism $\Phi : \mathcal{A} \to \mathcal{B}$ of complex associative algebras, for each $\gamma \in G$ we obtain another morphism $\Phi_\gamma : \mathcal{A} \to \mathcal{B}$in the obvious way: $\Phi_\gamma(f) \overset{def}{=} \Phi({}^\gamma f)$ (this is the induced action on $Hom(\mathcal{A}, \mathcal{B})$).

We start with the following parametric family of representations: for each positive prime p let $\Phi_p : \mathcal{A} \to \mathcal{B}$ given by

$$\Phi_p(a)(s,z)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}z^{v_p(n)} \qquad (3.1)$$

where *s* and *z* are complex variables and $v_p:\mathbb{N}\to\mathbb{Z}$ is the *p*-adic valuation (in Part I we give some detail for the convergence domain of these series). This is, indeed, a ring morphism of “Shapiro type” and we show some interesting examples, because we have not found any references in the literature. With a change of variables, (3.1) involves the Dirichlet and Bell series of the arithmetic function. We give here three simple and important examples:

1) (a) $\Phi_p(1)(s,z)=\sum_{n=1}^{\infty}\frac{z^{v_p(n)}}{n^s}=\frac{p^s-1}{p^s-z}\xi(s)$

(b) $\Phi_p(1)(s,xp^s)=\sum_{n=1}^{\infty}\frac{(xp^s)^{v_p(n)}}{n^s}=\frac{p^s-1}{p^s-xp^s}\xi(s)=\frac{1}{1-x}\left(1-\frac{1}{p^s}\right)\xi(s)$

where $\frac{1}{1-x}$ is the Bell series of the constant function = 1.

2) (a) $\Phi_p(\mu)(s,z)=\sum_{n=1}^{\infty}\mu(n)\frac{z^{v_p(n)}}{n^s}=\frac{p^s-z}{(p^s-1)\xi(s)}$

(b) $\Phi_p(\mu)(s,xp^s)=\sum_{n=1}^{\infty}\mu(n)\frac{(xp^s)^{v_p(n)}}{n^s}=\frac{p^s-xp^s}{(p^s-1)\xi(s)}=\frac{1-x}{\left(1-\frac{1}{p^s}\right)\xi(s)}$

where $1-x$ is the Bell series of the Moebius function μ.

3) (a) $\Phi_p(\lambda)(s,z)=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}z^{v_p(n)}=\frac{p^s+1}{p^s+z}\frac{\varsigma(2s)}{\varsigma(s)}$

(b) $\Phi_p(\lambda)(s,xp^s)=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}(xp^s)^{v_p(n)}=\frac{p^s+1}{p^s+xp^s}\frac{\varsigma(2s)}{\varsigma(s)}=\frac{1}{1+x}\left(1+\frac{1}{p^s}\right)\frac{\varsigma(2s)}{\varsigma(s)}$

where $\frac{1}{1+x}$ is the Bell series of the Liouville function λ.

Another feature of these representations that derivation with respect to z and evaluation at z = 0 leads to very interesting identities (see ***Part I***).

§9.3. The cohomology $H^{\bullet}(\mathcal{A},-)^{G}$ in the <u>associative</u> algebra $\mathcal{A}$

The cohomology we will use here is classical and its relationship with the Hochschild, Harrison and other cohomologies may be found in [Harrison] and [Robinson], for instance.

For an arbitrary $\mathcal{A}$-$\mathbb{C}$-bimodule M, each $C^{n}(\mathcal{A},M)$ is the set of the mappings

$$\omega : \overbrace{\mathcal{A}\times\mathcal{A}\times\ldots\times\mathcal{A}}^{n-times} \to M \tag{3.1}$$

(considered as abelian groups with pointwise addition). For n = 0, as usual, we have the isomorphism $C^{0}(\mathcal{A},M) = Hom(1,M) \cong M$. The coboundary operators are the mappings

$$\delta_{n} : C^{n}(\mathcal{A},M) \to C^{n+1}(\mathcal{A},M) \tag{3.2}$$

such that

$$\delta_{n}(\omega)(f_1, f_2,\ldots,f_{n+1}) =$$
$$= f_1.\omega(f_2, f_3,\ldots,f_{n+1}) + \sum_{i=1}^{n}(-1)^{i}\omega(f_1, f_{2,},\ldots,f_{i-1}, f_i * f_{i+1},\ldots,f_{n+1}) + (-1)^{n+1}\omega(f_1, f_2,\ldots,f_n).f_{n+1}$$

The verification of the fundamental identities $\delta_{n+1}\circ\delta_{n} = 0$ is routine (rather cumbersome, as usual: *Appendix* 1(a)) and the first coboundaries are:

. $\delta_0(x)(f) = f.x - x.f \quad (x \in M)$

. $\delta_1(\omega)(f_1, f_2) = f_1.\omega(f_2) - \omega(f_1 * f_2) + \omega(f_1).f_2$ (3.3)

. $\delta_2(\omega)(f_1, f_2, f_3) = f_1.\omega(f_2, f_3) - \omega(f_1 * f_2, f_3) + \omega(f_1, f_2 * f_3) - \omega(f_1, f_2).f_3$

Finally, the cocycles, boundaries and cohomological groups are, respectively, the usual ones: $Z^{n}(\mathcal{A},M) = Ker\delta_{n}$, $B^{n}(\mathcal{A},M) = IM\delta_{n}$ (for n = 0 is $B^{0}(\mathcal{A},M) = 0$), and

$H^n(\mathcal{A},M) = {Z^n(\mathcal{A},M)}\big/{B^n(\mathcal{A},M)}$. Observe that:

. $H^0(\mathcal{A},M) = Z^0(\mathcal{A},M) = \{x \in M : \forall f \in \mathcal{A} : f.x - x.f = 0\} = M^{\mathcal{A}}$;

[for the special case $M = \mathcal{A}$, with $f.x = f * x = x * f = x.f$, we have

$$H^0(\mathcal{A},\mathcal{A}) = Z^0(\mathcal{A},\mathcal{A}) = \{x \in \mathcal{A} : \forall f \in \mathcal{A} : f * x - x * f = 0\} = \mathcal{A} \ ]$$

. $Z^1(\mathcal{A},M) = \{\omega : \mathcal{A} \to M : \omega(f_1 * f_2) = f_1.\omega(f_2) + \omega(f_1).f_2\}$, i.e.: the 1-cocycles are the derivations $\partial : \mathcal{A} \to M$; (In [Shapiro] there is an explicit characterization of the derivations $\partial : \mathcal{A} \to \mathcal{A}$ that are continuous with respect a valuation norm).

. $B^1(\mathcal{A},M) = \{\omega : \mathcal{A} \to M : \exists x \in M : \omega(f) = f.x - x.f\}$: the 1-coboundaries are the *inner derivations*.

The general relationship between $H^2(\mathcal{A},M)$ with classes of extensions of $\mathcal{A}$ by M, [i.e., with exact sequences $0 \to M \overset{\alpha}{\to} W \overset{\beta}{\to} \mathcal{A} \to 0$, where W is a commutative algebra, β is an algebra morphism and α a module morphism such that $w.\alpha(x) = \alpha(\beta(w).x)$] may be found in [Robinson]. It will be not pursued here.

Now, let us transfer the group action in $\mathcal{A}$ to the cohomology.

Given a left action $G \times M \to M$, $(\gamma, x) \mapsto \gamma.x$ of G on M, there is a natural right action of G on $C^n(\mathcal{A},M)$: $\omega^\gamma := \gamma^{-1}.[\omega \circ (\gamma \times ... \times \gamma)]$, i.e:

$$(\omega^\gamma)(f_1,..., f_n) := \gamma^{-1}.\omega({}^\gamma f_1,...,{}^\gamma f_n) \qquad (3.4)$$

(It is a $\mathbb{C}$-linear action: see *Appendix* (b)).

***Lemma* 3.1**: Let G act on the $\mathcal{A}$-$\mathbb{C}$-bimodule M and let suppose that this is a left action compatible with the action of G in $\mathcal{A}$, i.e: $\gamma.(f.x) = {}^\gamma f.(\gamma.x)$ and $\gamma.(x.f) = (\gamma.x).{}^\gamma f$. (This is clearly true in the case $M = \mathcal{A}$, with $f.x = f * x = x * f = x.f$). Then: the coboundary operators commute with the action (3.4), i.e: $\delta_n(\omega^\gamma) = (\delta_n\omega)^\gamma$. Then the action passes to each $H^n(\mathcal{A},M)$ that is to say: $[\omega]^\gamma = [\omega^\gamma]$ is well defined (where $[\omega]$ is the cohomological class of ω.)

Proof of (i)

$$\delta_n(\omega^\gamma)(f_1, f_2,\dots, f_{n+1}) =$$

$$= f_1.(\omega^\gamma)(f_2, f_3,\dots, f_{n+1}) + \sum_{i=1}^{n}(-1)^i(\omega^\gamma)(f_1, f_{2,},\dots, f_{i-1}, f_i * f_{i+1},\dots, f_{n+1}) + (-1)^{n+1}(\omega^\gamma)(f_1, f_2,\dots, f_n).f_{n+1}$$

$$= f_1.[\gamma^{-1}.(\omega({}^\gamma f_2,{}^\gamma f_3,\dots,{}^\gamma f_{n+1})] +$$

$$+\sum_{i=1}^{n}(-1)^i[\gamma^{-1}.(\omega({}^\gamma f_1,{}^\gamma f_{2,},\dots,{}^\gamma f_{i-1},{}^\gamma f_i *{}^\gamma f_{i+1},\dots,{}^\gamma f_{n+1})] + (-1)^{n+1}[\gamma^{-1}.(\omega({}^\gamma f_1,{}^\gamma f_2,\dots,{}^\gamma f_n)]f_{n+1} =$$

$$=\gamma^{-1}.[{}^\gamma f_1.[(\omega({}^\gamma f_2,{}^\gamma f_3,\dots,{}^\gamma f_{n+1})] +$$

$$+\sum_{i=1}^{n}(-1)^i[\gamma^{-1}.(\omega({}^\gamma f_1,{}^\gamma f_{2,},\dots,{}^\gamma f_{i-1},{}^\gamma f_i *{}^\gamma f_{i+1},\dots,{}^\gamma f_{n+1})] + (-1)^{n+1}\gamma^{-1}.[(\omega({}^\gamma f_1,{}^\gamma f_2,\dots,{}^\gamma f_n).{}^\gamma f_{n+1}] =$$

$$=\gamma^{-1}.\{{}^\gamma f_1.(\omega({}^\gamma f_2,{}^\gamma f_3,\dots,{}^\gamma f_{n+1}) +$$

$$+\sum_{i=1}^{n}(-1)^i(\omega({}^\gamma f_1,{}^\gamma f_{2,},\dots,{}^\gamma f_{i-1},{}^\gamma f_i *{}^\gamma f_{i+1},\dots,{}^\gamma f_{n+1})] + (-1)^{n+1}(\omega({}^\gamma f_1,{}^\gamma f_2,\dots,{}^\gamma f_n).{}^\gamma f_{n+1}\} =$$

$$=(\delta_n\omega)^\gamma(f_1, f_2,\dots, f_n)\,\square$$

Let M be a commutative $\mathbb{C}$-algebra and let $\Phi : \mathcal{A} \to M$ be a morphism of $\mathbb{C}$-algebras. Then, M has the natural structure of $\mathcal{A}$-$\mathbb{C}$-bimodule with the operation $\mathcal{A}\times M \to M$

$$f.x = \Phi(f)x = x\,\Phi(f) = x.f$$

In this case we have

$$H^0(\mathcal{A},M) = Z^0(\mathcal{A},M) = \{x\in M : \forall f \in \mathcal{A} : \Phi(f)x - x\Phi(f) = 0\} = M$$
$$B^1(\mathcal{A}, M) = \{\omega : \mathcal{A} \to M : \exists x \in M : \omega(f) = \Phi(f)x - x\Phi(f)\} = 0$$
$$H^1(\mathcal{A},M) = Z^1(\mathcal{A},M) = \{\partial : \mathcal{A} \to M : \partial(f_1 * f_2) = \Phi(f_1)\partial(f_2) + \partial(f_1)\Phi(f_2)\}$$

If $\Phi : \mathcal{A} \to M$ is suryective: $\partial f = \Phi(x_f)$

$$\partial(f_1 * f_2) = \Phi(f_1)\partial(f_2) + \partial(f_1)\Phi(f_2)$$

$$\gamma.\Phi(f) = \Phi({}^\gamma f)$$

$$\gamma.(f.x) = \gamma.[\Phi(f)x] = {}^\gamma[\Phi(f)](\gamma.x) = \Phi({}^\gamma f)(\gamma.x)$$

and $\gamma.(x.f)=\gamma.(x\Phi(f)=(\gamma.x)[\gamma.\Phi(f)]=(\gamma.x)\Phi({}^{\gamma}f)$

$$\delta_1(\omega)(f_1,f_2)=f_1.\omega(f_2)-\omega(f_1*f_2)+\omega(f_1).f_2=\Phi(f_1)\omega(f_2)-\omega(f_1*f_2)+\omega(f_1)\Phi(f_2)$$

$$(\partial^{\gamma})(f)=\gamma^{-1}.\partial({}^{\gamma}f)$$

$$\partial({}^{\gamma}f_1*^{\gamma}f_2)=\Phi({}^{\gamma}f_1)\partial({}^{\gamma}f_2)+\partial({}^{\gamma}f_1)\Phi({}^{\gamma}f_2)$$

Appendix: Some cumbersome computations:

(a) We added the last factor $.\,f_{n+1}$ in the definition of the coboundaries, so we should prove that $\delta\circ\delta=0$. Here we go…

For n =1:

$$\delta_1\delta_0x(f_1,f_2)=f_1.\delta_0x(f_2)-\delta_0x(f_1*f_2)+\delta_0x(f_1).f_2=$$
$$=f_1.f_2.x-f_1.x.f_2-(f_1*f_2).x+x.(f_1*f_2)+f_1.x.f_2-x.f_1.f_2=0$$

For n = 2:

$$\delta_2\delta_1\omega(f_1,f_2,f_3)=f_1.\delta_1\omega(f_2,f_3)-\delta_1\omega(f_1*f_2,f_3)+\delta_1\omega(f_1,f_2*f_3)-\delta_1\omega(f_1,f_2).f_3=$$
$$=\overbrace{f_1.f_2.\omega(f_3)}^{1}-\overbrace{f_1.\omega(f_2*f_3)}^{2}+\overbrace{f_1.\omega(f_2).f_3}^{3}-$$
$$-\overbrace{(f_1*f_2).\omega(f_3)}^{1}+\overbrace{\omega(f_1*f_2*f_3)}^{4}-\overbrace{\omega(f_1*f_2).f_3}^{5}+$$
$$+\overbrace{f_1.\omega(f_2*f_3)}^{2}-\overbrace{\omega(f_1*f_2*f_3)}^{4}+\overbrace{\omega(f_1).(f_2*f_3)}^{6}-$$
$$-\overbrace{f_1.\omega(f_2).f_3}^{3}+\overbrace{\omega(f_1*f_2).f_3}^{5}-\overbrace{\omega(f_1).f_2.f_3}^{6}=0$$

(the terms with the same label cancel pairwise)

For n = 3:

$$\delta_3\delta_2\omega(f_1,f_2,f_3,f_4)=$$

$$=f_1.\delta_2\omega(f_2,f_3,f_4)-\delta_2\omega(f_1*f_2,f_3,f_4)+\delta_2\omega(f_1,f_2*f_3,f_4)-\delta_2\omega(f_1,f_2,f_3*f_4)+\delta_2\omega(f_1,f_2,f_3).f_4=$$

$$= \overbrace{f_1 \cdot f_2 . \omega(f_3, f_4)}^{1} - \overbrace{f_1 . \omega(f_2 * f_3, f_4)}^{2} + \overbrace{f_1 . \omega(f_2, f_3 * f_4)}^{3} - \overbrace{f_1 . \omega(f_2, f_3) \cdot f_4}^{4} +$$

$$- \overbrace{(f_1 * f_2) . \omega(f_3, f_4)}^{1} + \overbrace{\omega(f_1 * f_2 * f_3, f_4)}^{5} - \overbrace{\omega(f_1 * f_2, f_3 * f_4)}^{6} + \overbrace{\omega(f_1 * f_2, f_3) \cdot f_4}^{7} +$$

$$+ \overbrace{f_1 . \omega(f_2 * f_3, f_4)}^{2} - \overbrace{\omega(f_1 * f_2 * f_3, f_4)}^{5} + \overbrace{\omega(f_1, f_2 * f_3 * f_4)}^{8} + \overbrace{\omega(f_1, f_2 * f_3) \cdot f_4}^{9} +$$

$$- \overbrace{f_1 . \omega(f_2, f_3 * f_4)}^{3} + \overbrace{\omega(f_1 * f_2, f_3 * f_4)}^{6} - \overbrace{\omega(f_1, f_2 * f_3 * f_4)}^{8} + \overbrace{\omega(f_1, f_2) \cdot f_3 * f_4}^{10} +$$

$$+ \overbrace{f_1 . \omega(f_2, f_3) \cdot f_4)}^{4} - \overbrace{\omega(f_1 * f_2, f_3) \cdot f_4}^{7} + \overbrace{\omega(f_1, f_2 * f_3) \cdot f_4}^{9} + \overbrace{\omega(f_1, f_2) f_3 * f_4}^{10} = 0$$

(the terms with the same label cancel pairwise)(<u>skew-symmetry no needed</u>)

General proof:

$$\delta_n(\delta_{n-1}\omega)(f_1, f_2, ..., f_{n+1}) =$$

$$= f_1 . \delta_{n-1}\omega(f_2, f_3, ..., f_{n+1}) - \delta_{n-1}\omega(f_1 * f_2, f_3, ..., f_{n+1}) + \delta_{n-1}\omega(f_1, f_2 * f_3, ..., f_{n+1}) -$$

$$... + (-1)^n \delta_{n-1}\omega(f_1, f_2, ..., f_n * f_{n+1}) + (-1)^{n+1} \delta_{n-1}\omega(f_1, f_2, ..., f_n) . f_{n+1} =$$

$$= \overbrace{f_1 \cdot f_2 . \omega(f_3, ..., f_{n+1})}^{1 \otimes 1} - \overbrace{f_1 . \omega(f_2 * f_3, ..., f_{n+1})}^{1 \otimes 2} + ... + \overbrace{(-1)^{n-1} f_1 . \omega(f_2, ..., f_{n-1} * f_{n+1})}^{1 \otimes n} + \overbrace{(-1)^n f_1 . \omega(f_2, ..., f_n) \cdot f_{n+1}}^{1 \otimes n+1} +$$

$$- \overbrace{(f_1 * f_2) . \omega(f_3, ..., f_{n+1})}^{2 \otimes 1} + \overbrace{\omega(f_1 * f_2 * f_3, ..., f_{n+1})}^{2 \otimes 2} + ... + \overbrace{(-1)^{n-1} \omega(f_1 * f_2, ..., f_{n-1} * f_{n+1})}^{2 \otimes n} + \overbrace{(-1)^n \omega(f_1 * f_2, ..., f_n) \cdot f_{n+1}}^{2 \otimes n+1} +$$

$$+ \overbrace{f_1 . \omega(f_2 * f_3, ..., f_{n+1})}^{3 \otimes 1} - \overbrace{\omega(f_1 * f_2 * f_3, ..., f_{n+1})}^{3 \otimes 2} + ... + \overbrace{(-1)^{n-1} \omega(f_1, f_2 * f_3, ..., f_n * f_{n+1})}^{3 \otimes n} + \overbrace{(-1)^n \omega(f_1, f_2 * f_3, ..., f_n) . f_{n+1}}^{3 \otimes n+1} +$$

….

$$+ \overbrace{(-1)^{n+1} f_1 . \omega(f_2, ,..., f_n * f_{n+1})}^{n+1 \otimes 1} - \overbrace{(-1)^{n+1} \omega(f_1 * f_2, ,..., f_n * f_{n+1})}^{n+1 \otimes 2} + ... + \overbrace{(-1)^{n+1+n-1} \omega(f_1, .., f_{n-1} * f_n * f_{n+1})}^{n+1 \otimes n} + \overbrace{(-1)^{n+1+n} \omega(f_1, ,..., f_{n-1}) . (f_n * f_{n+1})}^{n+1 \otimes n+1} +$$

$$+\overbrace{(-1)^{n+2} f_1.\omega(f_2,,\ldots,f_n).f_{n+1}}^{n+2\otimes 1}-\overbrace{(-1)^{n+2}\omega(f_1 * f_2,,\ldots,f_n).f_{n+1}}^{n+1\otimes 2}+\ldots+\overbrace{(-1)^{n+2+n-1}\omega(f_1,..,f_{n-1} * f_n).f_{n+1}}^{n+2\otimes n}+\overbrace{(-1)^{n+2+n}\omega(f_1,,\ldots,f_{n-1}).(f_n * f_{n+1})}^{n+2\otimes n+1}$$

(the tensor symbol is a joke…): terms cancel pairwise: 1 ⊗ 1 with 2 ⊗ 1, 1 ⊗ 2 with 3 ⊗ 1, …, 1 ⊗ *n* with *n*+1 ⊗ 1, 1 ⊗ *n*+1 with *n*+2 ⊗ 1; 2 ⊗ 2 with 3 ⊗ 2, …, 2 ⊗ *n* with *n*+1 ⊗ 2 , 2 ⊗ *n*+1 with *n*+1 ⊗ 2; ….; *n*+1 ⊗ *n* with *n*+2 ⊗ *n* and *n*+1 ⊗ *n*+1 with *n*+2 ⊗ *n*+1.)
(skew-symmetry no needed)□

(b) Verification that (3.4) define a $\mathbb{C}$ linear action of *G* on each $C^n(\mathcal{A}, M)$:

P/ The linearity of the mappings $\omega \mapsto \omega^{\gamma}$ is obvious, so is the identity $\omega^{1_G} = \omega$. Now:

$$(\omega^{\gamma})^{\sigma}(f_1,\ldots,f_n) := \sigma^{-1}.(\omega^{\gamma})({}^{\sigma}f_1,\ldots,{}^{\sigma}f_n) = \sigma^{-1}.\gamma^{-1}.\omega[{}^{\gamma}({}^{\sigma}f_1),\ldots,{}^{\gamma}({}^{\sigma}f_n)] =$$
$$= (\gamma \circ \sigma)^{-1}.\omega({}^{\gamma\circ\sigma}f_1,\ldots,{}^{\gamma\circ\sigma}f_n) = (\omega^{\gamma\circ\sigma})(f_1,\ldots,f_n)$$

Another way:

$$(\omega^{\gamma})^{\sigma} = \{\gamma^{-1}.[\omega \circ (\gamma \times \ldots \times \gamma)]\}^{\sigma} = \sigma^{-1}.\gamma^{-1}.[\omega \circ (\gamma \times \ldots \times \gamma) \circ (\sigma \times \ldots \times \sigma)] =$$
$$= (\gamma \circ \sigma)^{-1}.[\omega \circ (\gamma \circ \sigma \times \ldots \times \gamma \circ \sigma)] = \omega^{\gamma\circ\sigma}$$

□